Neural-Network-Based Fully Distributed Adaptive Consensus for a Class of Uncertain Multiagent Systems

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***Abstract*— In this article, we revisit the problem of distributed neuroadaptive consensus for uncertain multiagent systems (MASs) in the presence of unmodeled nonlinearities as well as unknown disturbances. Robust consensus controllers comprising a linear feedback term, a discontinuous feedback term, and a neural network approximation term are constructed, where in each term, the weight part is endowed with some dynamical changing law. The asymptotic convergence of the consensus errors is theoretically proved based on the graph theory, nonsmooth analysis, and Barbalat's lemma. Both leaderless consensus and leader–follower tracking problems are considered before the results are further extended to containment problem in the presence of multileaders. A dramatic feature of the proposed method, in comparison with related works, is the fully distributed fashion of the information, requiring neither the underlying Laplacian eigenvalues nor the input upper bounds of the leaders (if exist). Several numerical examples are presented to testify the theoretical results.**

***Index Terms*— Consensus, distributed adaptive control, neural networks (NNs), nonlinear multiagent system (MAS).**

I. Introduction

THERE has been a significant attention on the cooperative control of multiagent systems (MASs) over the past two decades from various scientific communities, due to its applicability in different fields such as traffic management [1], power systems [2], autonomous robots [3], and distributed optimization [4], [5], to name a few. Typically, the study of cooperative control of MASs devotes to analyze the emergence of various global behaviors, such as consensus [6], flocking [7], and formation [8], based on local communications among the individuals.

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Consensus problems, concerning how to reach an agreement, can be traced back to the seminal work of [9] and [10], where the concept of consensus was first introduced in [9], and a discrete-time model known as the Vicseck model was built to study a group of autonomous agents with the same magnitude of speed but different headings under noise perturbation in [10]. Since the new century, consensus problems have been persistently addressed in the literature and many profound results have been established [11]–[13]. According to whether there is a reference agent in the MAS, the formulation of consensus in the literature can be roughly divided into leaderless consensus and leader–follower consensus (which is also referred to as consensus tracking). For the first category, agents reach a common trajectory that is prior unknown to all individuals. For the second category, there is a reference agent called the leader, on which the rest of agents (followers) cooperatively follow up in time. As for the containment control with multiple leaders [14]–[17], the followers seek to move into the convex hull spanned by the leaders. This vivid cooperative behavior is valuable in some specific tasks. For instance, by equipping several autonomous robots (leaders) with necessary sonar devices to detect the hazardous obstacles, safe obstacle avoidance task of the whole group can be accomplished by employing containment controllers for the rest robots (followers) [14].

One distinguishing feature of the multiagent setup is the distribution of information, allowing agents to maintain their local variables and communicate with their neighbors before finishing computations locally. There are at least two advantages of distributed control strategies compared with centralized ones, i.e., cost saving and privacy protection. Specifically, centralized controllers based on global information are often costly or even impossible to implement in many network systems, especially in large-scale ones, due to the limited sensing abilities of sensors. Moreover, opening all information to each individual is insecure from the manager's perspective. Therefore, distributed strategies for MASs have been consistently pursued in recent years [18]–[26]. In [18], distributed adaptive dynamic consensus protocol from both edge- and node-based perspectives is designed for general linear MASs, where the edge-based ideas are further utilized to coordinate the Lipschitz nonlinear MASs in [19], whereas the node-based counterparts are extended to handle the consensus tracking problem in [20]. An active–passive MAS that consists of active agents (subject to exogenous inputs) and passive agents (without any inputs) is concerned in [21], where an integral action-based distributed control approach, which, in a sense analogous to the node-based adaptive strategies, is proposed to realize the average consensus to the exogenous inputs applied to the active agents. In [22], the edge-based distributed adaptive consensus protocol is constructed for one-sided Lipschitz and quadratic inner-boundedness nonlinear MASs with and without external disturbances. Distributed robust containment control of linear MASs subject to time-varying uncertainties is addressed in [23], where the uncertainties of the system are compensated with a discontinuous term and fully distributed fashion is achieved with adaptive coupling weights. Fully distributed event-triggered protocols are formulated for linear MASs with or without matched uncertainties in [24] and [25], respectively, which both admit edge-based frameworks. In [26], fully distributed leaderless and leader–follower consensus problems are considered for linear and Lipschitz nonlinear MASs via directed spanning tree-based adaptive control. It can be seen that adaptive control serves indeed as a powerful tool toward full distribution, and also, it should be emphasized that fully distributed schemes have the superiority in which the designers need not pay attention to any global information such as the underlying graph Laplacian matrix as long as some connectivity conditions are satisfied by the target MASs, which are more flexible than classical ones. Note that in another closely related topic, which is the synchronization of complex networks, the implementation of distributed adaptive controllers is also arousing many interests (see [27]–[30]).

In most of the aforementioned references, however, the internal dynamics of the MASs are deterministic. For the asymptotical tracking problem of high-order nonlinear MASs with time-varying reference subjected to unknown parameters and uncertain disturbances, a novel backstepping smooth distributed adaptive control scheme is proposed in [31]. By further adopting backstepping technique and proposing a novel Nussbaum-type function, fully distributed adaptive laws are designed for high-order systems with unknown control directions as well as parameter uncertainties in [32]. Moreover, results considering unmodeled dynamics have recently appeared in [33]–[37], where neural networks (NNs) show their extraordinary talent in approximating the unknown nonlinearities. In [33], neuroadaptive consensus tracking protocols for unknown nonlinear MASs are designed, which allow each agent to adjust the local NN weights in a distributed fashion. Later in [34], tracking problem of uncertain MASs with unknown matched nonlinearities as well as disturbances is considered with state- and observer-based protocols, respectively. Consensus control of a class of nonlinear time-delay MASs is addressed in [35] by using radial basis function NNs. For highly nonlinear MASs, a distributed neuroadaptive tracking control with prescribed performance is proposed in [36]. A common feature of these works, and many other NN-based works, is that the tracking errors are uniformly ultimately bounded (UUB) theoretically. Recently, in [37], asymptotic neuroadaptive containment control of MASs with unmodeled dynamics has been realized by defining a novel pseudoideal approximating weight matrix. Nevertheless, the results of handling unmodeled dynamics in these aforementioned contributions are not fully distributed as either the underlying Laplacian matrix of the concerned MASs or the input upper bound(s) of the leader(s), which are the global information, is required in designing the protocols.

Motivated by the above discussions, this article intends to solve the node-based adaptive consensus (tracking) as well as containment problems for a class of uncertain MASs with partially unmodeled dynamics and external disturbances. The main difficulty lies in that one must guarantee these global behaviors and compensate for the unknown nonlinearities at the same time, without knowing any global information such as the eigenvalues of the underlying graph Laplacian matrix. To effectively compensate for the uncertainties arousing from the unknown nonlinearities and disturbances, robust controllers comprising a linear term, a discontinuous term, and an NN approximating term are proposed. With the help of Barbalat's lemma and nonsmooth analysis, fully distributed cooperative schemes are presented and the asymptotical convergence for each scheme is theoretically proved. For the first time (to our best knowledge), we show that the adaptive coupling weight technique and the NN adaptive technique can work together. The main contributions of this article are threefold.

1) In comparison with related works in [19], [20], and [26], where only homogeneous linear or Lipschitz nonlinear MASs were involved, a more general model consisting of heterogeneous uncertain nonlinear agents is considered. Heterogeneous disturbances and nonzero control inputs of the leader agents (if exist) are also modeled. The consideration of these uncertainties extends the serviceable range of the model.
2) Different from [33]–[36], where the consensus tracking errors were governed to be UUB due to the presence of unmodeled dynamics, asymptotic consensus and tracking are realized by introducing nonsmooth techniques, and the results are then extended to the containment case with multiple leaders in a systematic way.
3) In comparison with [34] and [37], fully distributed paradigms for consensus-based control of the uncertain dynamical MASs are developed by introducing adaptive strategies in the coupling weights, which enhances the flexibility for control and convenience for maintenance.

The remaining of this article is outlined as follows. In Section II, some preliminaries on graph theory and nonsmooth analysis are presented, followed by the model description. In Section III, leaderless consensus problem via distributed neuroadaptive control is handled. Next, the consensus tracking problem with a leader of possibly nonzero input is addressed in Section IV, where the results are then generalized into the case with multiple leaders. In Section V, several numerical examples are given to testify the theoretical results. Section VI finally concludes this article and discusses some future interests.

Notations: $\mathbb{R}$ ($\mathbb{R}^+$), $\mathbb{R}^n$, and $\mathbb{R}^{n\times p}$ denote the sets of real (positive) numbers, n-dimensional real vectors, and $n \times p$ real matrices, respectively. Denote by $\boldsymbol{O}_n$ ($\boldsymbol{I}_n$) the $n \times n$ zero (identity) matrix, and $\mathbf{0}_n(\mathbf{1}_n)$ the n-dimensional column vector with all elements being zero (one). Let $\|\cdot\|$ ($\|\cdot\|_F$) denote the Euclidian (Frobenius) norm of a vector (matrix). Moreover, the 1-norm and ∞-norm of a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ are, respectively, denoted by $\|x\|_1$ and $\|x\|_\infty$, i.e., $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_{i\in\{1,\ldots,n\}} |x_i|$. The matrix inequality $A > (\geq) 0$ means that A is positive (semi)definite. Suppose that all the eigenvalues of A are real, and denote by $\lambda_{\min}(A)$ the smallest one. The trace of a square matrix A is denoted by $\mathrm{tr}(A)$. Symbol $\otimes$ represents the Kronecker product. For a set of vectors (matrices) $x_1, x_2, \ldots, x_n$, $\mathrm{col}(x_1, x_2, \ldots, x_n) = (x_1{}^T, x_2{}^T, \ldots, x_n{}^T)^T$ is the column vectorization and $\mathrm{diag}(x_1, x_2, \ldots, x_n)$ is the diagonal matrix with diagonal elements $x_1, x_2, \ldots, x_n$. The abbreviation $\mathrm{sgn}(\cdot)$ is the signum function defined componentwise. The term "iff" represents "if and only if."

II. Preliminaries and Model Description

A. Graph Theory

A directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ or $\mathcal{G}$, if no conflicts will arise, is specified by a node set $\mathcal{V} = \{1, 2, \ldots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, in which i is a neighbor of j for an edge (i, j). A path on $\mathcal{G}$ from vertex υ_1 to vertex υ_s corresponds to a sequence of ordered edges of the form $(\upsilon_k, \upsilon_{k+1}), k = 1, 2, \ldots, s-1$. A digraph is said to contain a directed spanning tree, if there exists a node called the root, which has no neighbors, such that one can find a unique path from the node to every other node. $\mathcal{G}$ is said to be undirected if $(i, j) \in \mathcal{E}$ iff $(j, i) \in \mathcal{E}$. An undirected graph is said to be connected if there exists a path between each pair of distinct nodes. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $\mathcal{G}$ defined by $a_{ii} = 0$ and $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = (l_{ij}) \in \mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as: $l_{ij} = -a_{ij}, i \neq j$, and $l_{ii} = \sum_{k=1,k\neq i}^N a_{ik}, i = 1, 2, \ldots, N$. In this article, $\mathcal{G}$ is regarded as a simple graph, for which multiple edges and self-loops are not permitted.

Lemma 1 [38]: The Laplacian matrix $\mathcal{L}$ has an eigenvalue 0 with $\mathbf{1}_N$ as a corresponding right eigenvector, and all the other eigenvalues have positive real parts. Furthermore, 0 is a simple eigenvalue of $\mathcal{L}$ iff $\mathcal{G}$ has a directed spanning tree.

B. Nonsmooth Theory

Consider the following vector differential equation with a discontinuous right-hand side:

$$\dot{z} = g(z, t) \tag{1}$$

where $g : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded. A vector function $z(t)$ is called a Filippov solution of differential system (1) over $[t_a, t_b]$, if $z(t)$ is absolutely continuous on $[t_a, t_b]$ and satisfies, for almost all t, the differential inclusion that $\dot{z} \in \mathcal{K}[g](z, t)$. Here, $\mathcal{K}[g](z, t) \triangleq \bigcap_{\alpha>0} \bigcap_{\mu(\bar{N})=0} \bar{co}\{g(B(z, \alpha) - \bar{N}, t)\}$, in which $\bigcap_{\mu(\bar{N})=0}$ represents the intersection over all sets $\bar{N}$ of Lebesgue measure zero, $\bar{co}\{\cdot\}$ is the convex closure operator, and $B(z, \alpha)$ denotes the open ball of radius α centered at z.

Suppose that $V(z) : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, and then, the set-valued Lie derivative of $V(z)$ along system (1) is defined as

$$\dot{\tilde{V}}(z(t)) = \bigcap_{\xi\in\partial V(z(t))} \xi^T \mathcal{K}[g](z(t), t)$$

where the symbol $\partial V(z)$ represents Clarke's generalized gradient of V at z defined as $\partial V(z) = \bar{co}\{\lim_{i\to\infty} \nabla V(z_i) | z_i \to z, z_i \notin \Omega_u \bigcup \bar{N}\}$. Here, Ω_u denotes the set of Lebesgue measure zero where ∇V does not exist and $\bar{N}$ is an arbitrary set of Lebesgue measure zero.

Lemma 2 (Barbalat's Lemma [39]): Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a uniformly continuous function. Suppose that $\lim_{t\to\infty} \int_0^t h(\tau)d\tau$ exists and is finite, and then, $h(t) \to 0$ as $t \to \infty$.

C. Model Description

The concerned MAS consists of N agents with heterogeneous matching nonlinear dynamics subject to external disturbances. Concretely, the dynamics of the ith agent can be modeled by

$$\dot{x}_i = Ax_i + B(u_i + f_i(x_i) + d_i(t)) \tag{2}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ are the state and control input of agent i, respectively. The matrices A and B are known, constant, and with compatible dimensions such that (A, B) is stabilizable. The function $f_i(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is the unknown but smooth matching nonlinearity on the actuator side, and $d_i(t) \in \mathbb{R}^m$ denotes the bounded matching disturbance of agent i such that

$$\|d_i(t)\|_\infty \leq d_{iM}. \tag{3}$$

Note that the agents are essentially heterogeneous in considering that the lumped uncertainties $f_i(x_i)$ and $d_i(t)$ are allowed to be different. Besides, the constant d_{iM} could be unknown to the control designer. In this article, NNs are employed to directly compensate for the unknown nonlinear functions in the system. Toward this, the following assumption is needed.

Assumption 1 [34]: The unknown matching function $f_i(x_i)$ for agent i in (2) can be linearly parameterized by an NN as

$$f_i(x_i) = W_i^T S_i(x_i) + \epsilon_i \quad \forall x_i \in \Omega_i \tag{4}$$

where $W_i \in \mathbb{R}^{s\times m}$ is an ideal weight matrix which is unknown but bounded by $\|W_i\|_F \leq W_{iM}$ and $S_i(x_i) : \mathbb{R}^n \mapsto \mathbb{R}^s$ is a vector collection of basis functions of the form $S_i(x_i) = (S_{i1}(x_i), \ldots, S_{is}(x_i))^T$ and satisfies $\|S_i\| \leq S_{iM}$. ϵ_i is the approximation error vector satisfying $\|\epsilon_i\| \leq \epsilon_{iM}$ and Ω_i is a sufficiently large compact set.

Remark 1: Assumption 1 can be guaranteed by the Stone–Weierstrass theorem [40], which is fairly standard when processing approximation tasks with NNs [33], [36], [37], [41]. Here, the constants W_{iM}, S_{iM}, and ϵ_{iM} could be unknown for the controller design, which serve only for theoretical analysis.

Remark 2: It is worth mentioning that our perspective is different from that of [42] and [43], where some boundness assumptions are made regarding the unknown matched nonlinearities. In our case, the approximation abilities of NNs are borrowed to directly compensate for the nonlinear uncertainties without assuming their boundness. Here, f is not allowed to depend explicitly on time, though this does not hurt the generality since one can easily turn the nonautonomous equations into autonomous ones by introducing another state variable $x_{n+1} = t$ [44]. Besides, the considered model is closely related, albeit different from that of [34] as all three agent networks, i.e., leaderless MAS, leader–follower MAS, and multileader MAS, will be considered.

III. Leaderless Consensus via Distributed Neuroadaptive Control

In this section, the general consensus problem of (2) is concerned. The control object is to design suitable u_i and NN weight W_i for each agent to realize asymptotic consensus in (2), i.e., $\lim_{t\to\infty} \|x_i - x_j\| = 0, \forall i, j \in \mathcal{V}$.

Assumption 2: The communication topology $\mathcal{G}$ among the N agents is undirected and connected.

To keep the notations simple, define the global consensus error and the local error measurement of agent i as $e_i = x_i - (1/N)\sum_{j=1}^{N} x_j$ and $\delta_i = \sum_{j=1}^{N} a_{ij}(x_i - x_j)$, respectively. Let $x = \mathrm{col}(x_1, x_2, \ldots, x_N)$, $e = \mathrm{col}(e_1, e_2, \ldots, e_N)$, $\delta = \mathrm{col}(\delta_1, \delta_2, \ldots, \delta_N)$ and $\Xi = \boldsymbol{I}_N - (1/N)\mathbf{1}\mathbf{1}^T$. Then, we have $e = (\Xi \otimes \boldsymbol{I}_n)x$ and $\delta = (\mathcal{L} \otimes \boldsymbol{I}_n)e$. Note that Ξ has a simple eigenvalue 0 and $N-1$ multiple eigenvalue 1 and there holds under Assumption 2 that $\mathcal{L}\Xi = \Xi\mathcal{L} = \mathcal{L}$.

Let $\alpha_i(t)$ and $\beta_i(t)$ be the dynamic coupling strengths for the direct feedback effects and signed feedback effects, respectively, between agent i and its neighbors. Besides, introduce $\hat{W}_i(t)$ as the estimation of the unknown weight matrix W_i and $\overline{W}_i(t)$ as the pseudoideal weight matrix for the NN approximator of agent i. Then, based on the relative states of neighboring agents, the dynamic protocols for agents i, $i \in \mathcal{V}$, are designed as

$$\begin{aligned} u_i &= \alpha_i(t)F\delta_i + \beta_i(t)\mathrm{sgn}(F\delta_i) - \hat{W}_i^T(t)S_i(x_i) \\ \dot{\alpha}_i &= \kappa_i\delta_i^T\Gamma\delta_i \\ \dot{\beta}_i &= \nu_i\|F\delta_i\|_1 \end{aligned} \tag{5}$$

with NN adaptive law

$$\begin{aligned} \dot{\hat{W}}_i &= \tau_i\big[S_i(x_i)\delta_i^T P^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i(t))\big] \\ \dot{\overline{W}}_i &= \sigma_i\pi_i(\hat{W}_i - \overline{W}_i) \end{aligned} \tag{6}$$

where F and Γ are feedback gain matrices to be designed, $\kappa_i, \nu_i, \tau_i, \sigma_i$, and π_i are positive scalars, and $P > 0$ remains to be selected.

Remark 3: The proposed controller u_i in (5) consists of three parts: $\alpha_i(t)F\delta_i$ is the feedback term to drive all the agents to a common trajectory, $\beta_i(t)\mathrm{sgn}(F\delta_i)$ is the discontinuous feedback term to eliminate the influences of unknown disturbances as well as NN approximation errors, and $\hat{W}_i^T(t)S_i(x_i)$ is the NN-based compensation term to the unknown nonlinearities of agent i.

It is clear that $e = 0$ iff $x_1 = \cdots = x_N$, i.e., the consensus achieves. Thus, we turn to the convergence analysis of e. Define $\mathcal{M}_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, $\mathcal{M}_2 = \mathrm{diag}(\beta_1, \ldots, \beta_N)$, $\tilde{W}_i = \hat{W}_i - W_i$ where $i = 1, \ldots, N$, $\tilde{W} = \mathrm{diag}(\tilde{W}_1, \ldots, \tilde{W}_N)$, $S(x) = \mathrm{col}(S_1(x_1), \ldots, S_N(x_N))$, $\epsilon = \mathrm{col}(\epsilon_1, \ldots, \epsilon_N)$, and $d = \mathrm{col}(d_1, \ldots, d_N)$. Then, based on (2), (4), and (5), we can obtain the dynamics of error system e as

$$\begin{aligned} \dot{e} = &(\boldsymbol{I}_N \otimes A + \Xi\mathcal{M}_1\mathcal{L} \otimes BF)e \\ &+ (\Xi\mathcal{M}_2 \otimes B)\mathrm{sgn}((\mathcal{L} \otimes F)e) \\ &- (\Xi \otimes B)(\tilde{W}^T S(x) - \epsilon - d). \end{aligned} \tag{7}$$

By Assumption 1, there exist positive constants W_M, S_M, and ϵ_M such that

$$\|W\|_F \le W_M, \quad \|S(x)\|_F \le S_M, \quad \|\epsilon\|_\infty \le \epsilon_M. \tag{8}$$

Also, by (3), there exists $d_M > 0$ such that $\|d\|_\infty \le d_M$.

Now, we are ready to give the main theorem of this section.

Theorem 1: Under Assumptions 1 and 2, the asymptotic consensus problem of MAS (2) can be solved under distributed adaptive protocol (5) and NN weight adaptive law (6) with $F = -B^T P^{-1}$, $\Gamma = P^{-1}BB^T P^{-1}$, and scalars κ_i, ν_i, τ_i, and σ_i $\pi_i \in \mathbb{R}^+$. Here, $P > 0$ is a solution of the following LMI:

$$AP + PA^T - \eta BB^T + \theta P \le 0 \tag{9}$$

for scalars $\eta, \theta \in \mathbb{R}^+$. Moreover, each estimated NN weight matrix $\hat{W}_i$ converges to the corresponding pseudoideal weight matrix $\overline{W}_i$, i.e., $\lim_{t\to\infty}(\hat{W}_i - \overline{W}_i) = \boldsymbol{O}$ and each coupling gain $\alpha_i(t)$ as well as $\beta_i(t)$ converges to some finite value.

Proof: Define $\tilde{\overline{W}}_i = \overline{W}_i - W_i$, $i = 1, \ldots, N$, and consider the following Lyapunov function candidate:

$$\begin{aligned} V_1 = &e^T(\mathcal{L} \otimes P^{-1})e \\ &+ \sum_{i=1}^{N}\frac{1}{\kappa_i}(\alpha_i(t) - \bar{\alpha})^2 + \sum_{i=1}^{N}\frac{1}{\nu_i}(\beta_i(t) - \bar{\beta})^2 \\ &+ \sum_{i=1}^{N}\mathrm{tr}\left(\frac{1}{\tau_i}\tilde{W}_i^T\tilde{W}_i\right) + \sum_{i=1}^{N}\mathrm{tr}\left(\frac{1}{\pi_i}\tilde{\overline{W}}_i^T\tilde{\overline{W}}_i\right) \end{aligned} \tag{10}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are two positive constants to be determined later and P is a solution of (9).

Note that the right-hand side of (7) is discontinuous, and the stability of e will be analyzed by using differential inclusions and nonsmooth theory. Since the signum function is measurable and essentially bounded, the solution for (7) exists in the sense of Filippov. By using terms of differential inclusions, (7) is written as

$$\begin{aligned} \dot{e} \in^{a.e.} &\mathcal{K}[(\boldsymbol{I}_N \otimes A + \Xi\mathcal{M}_1\mathcal{L} \otimes BF)e \\ &+ (\Xi\mathcal{M}_2 \otimes B)\mathrm{sgn}((\mathcal{L} \otimes F)e) \\ &- (\Xi \otimes B)(\tilde{W}^T S(x) - \epsilon - d)] \end{aligned} \tag{11}$$

where $a.e.$ represents "almost everywhere". Then, by using the properties of $\mathcal{K}[\cdot]$, the set-valued Lie derivative of V_1 along (7) can be obtained as

$$\begin{aligned} \dot{\tilde{V}}_1 = &2e^T(\mathcal{L} \otimes P^{-1}A + \mathcal{L}\mathcal{M}_1\mathcal{L} \otimes P^{-1}BF)e \\ &+ \sum_{i=1}^{N}\frac{2}{\kappa_i}(\alpha_i(t) - \bar{\alpha})\dot{\alpha}_i + \sum_{i=1}^{N}\frac{2}{\nu_i}(\beta_i(t) - \bar{\beta})\dot{\beta}_i \\ &- 2e^T(\mathcal{L} \otimes P^{-1}B)(\tilde{W}^T S(x) - \epsilon - d) \\ &+ 2\sum_{i=1}^{N}\mathrm{tr}\left(\frac{1}{\tau_i}\tilde{W}_i^T\dot{\tilde{W}}_i\right) + 2\sum_{i=1}^{N}\mathrm{tr}\left(\frac{1}{\pi_i}\tilde{\overline{W}}_i^T\dot{\tilde{\overline{W}}}_i\right) \\ &+ 2\mathcal{K}[e^T(\mathcal{L}\mathcal{M}_2 \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L} \otimes F)e)]. \end{aligned} \tag{12}$$

Substituting $\dot{\alpha}_i$ and $\dot{\beta}_i$ in (5), $\dot{\hat{W}}_i$ and $\dot{\overline{W}}_i$ in (6) into (12) gives

$$\begin{aligned}\dot{\tilde{V}}_1 = {} & 2e^T(\mathcal{L}\otimes P^{-1}A + \mathcal{L}\mathcal{M}_1\mathcal{L}\otimes P^{-1}BF)e \\ & + 2\sum_{i=1}^{N}(\alpha_i(t)-\bar{\alpha})\delta_i^T\Gamma\delta_i + 2\sum_{i=1}^{N}(\beta_i(t)-\bar{\beta})\|F\delta_i\|_1 \\ & - 2e^T(\mathcal{L}\otimes P^{-1}B)(\tilde{W}^T S(x) - \epsilon - d) \\ & + 2\sum_{i=1}^{N}\mathrm{tr}\big(\tilde{W}_i^T\big(S_i(x_i)\delta_i^T P^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i)\big)\big) \\ & + 2\sum_{i=1}^{N}\mathrm{tr}\big(\sigma_i\tilde{\overline{W}}_i^T(\hat{W}_i - \overline{W}_i)\big) \\ & + 2\mathcal{K}[e^T(\mathcal{L}\mathcal{M}_2\otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}\otimes F)e)]. \end{aligned} \tag{13}$$

Since $F = -B^T P^{-1}$, $\Gamma = P^{-1}BB^TP^{-1}$, some mathematical manipulations give that

$$\begin{aligned} & e^T(\mathcal{L}\mathcal{M}_1\mathcal{L}\otimes P^{-1}BF)e \\ & \quad = -e^T(\mathcal{L}\otimes \boldsymbol{I}_n)(\mathcal{M}_1\otimes P^{-1}BB^TP^{-1})(\mathcal{L}\otimes \boldsymbol{I}_n)e \\ & \quad = -\delta^T(\mathcal{M}_1\otimes\Gamma)\delta \\ & \quad = -\sum_{i=1}^{N}\alpha_i(t)\delta_i^T\Gamma\delta_i \end{aligned} \tag{14}$$

$$\begin{aligned} & e^T(\mathcal{L}\mathcal{M}_2\otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}\otimes F)e) \\ & \quad = -\delta^T(\mathcal{M}_2\otimes F^T)\mathrm{sgn}((\boldsymbol{I}_N\otimes F)\delta) \\ & \quad = -\sum_{i=1}^{N}\beta_i(t)\delta_i^TF^T\mathrm{sgn}(F\delta_i) \\ & \quad = -\sum_{i=1}^{N}\beta_i(t)\|F\delta_i\|_1 \end{aligned} \tag{15}$$

where we have used the fact that $x^T\mathrm{sgn}(x) = \|x\|_1$ for an arbitrary real column vector x to get the second equality. Since $\mathcal{K}[f] = f$ for an arbitrary continuous function f, then it follows from (13)–(15) that:

$$\begin{aligned}\dot{\tilde{V}}_1 = {} & 2e^T(\mathcal{L}\otimes P^{-1}A)e \\ & - 2\bar{\alpha}\sum_{i=1}^{N}\delta_i^T\Gamma\delta_i - 2\bar{\beta}\sum_{i=1}^{N}\|F\delta_i\|_1 \\ & - 2e^T(\mathcal{L}\otimes P^{-1}B)(\tilde{W}^TS(x) - \epsilon - d) \\ & + 2\sum_{i=1}^{N}\mathrm{tr}\big(\tilde{W}_i^T\big(S_i(x_i)\delta_i^TP^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i)\big)\big) \\ & + 2\sum_{i=1}^{N}\mathrm{tr}\big(\sigma_i\tilde{\overline{W}}_i^T(\hat{W}_i - \overline{W}_i)\big). \end{aligned} \tag{16}$$

It can be seen that the set-valued Lie derivative of $V_1(t)$ is actually a singleton in this case.

Note that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ holds for any compatible matrices X, Y, one has

$$e^T(\mathcal{L}\otimes P^{-1}B)(\tilde{W}^TS(x)) = \sum_{i=1}^{N}\mathrm{tr}\big(\tilde{W}_i^TS_i(x_i)\delta_i^TP^{-1}B\big). \tag{17}$$

Then

$$\begin{aligned}\dot{\tilde{V}}_1 = {} & 2e^T(\mathcal{L}\otimes P^{-1}A)e \\ & - 2\bar{\alpha}\sum_{i=1}^{N}\delta_i^T\Gamma\delta_i - 2\bar{\beta}\sum_{i=1}^{N}\|F\delta_i\|_1 \\ & + 2e^T(\mathcal{L}\otimes P^{-1}B)(\epsilon + d) \\ & - 2\sum_{i=1}^{N}\mathrm{tr}(\sigma_i(\hat{W}_i - \overline{W}_i)^T(\hat{W}_i - \overline{W}_i)). \end{aligned} \tag{18}$$

Let $\bar{e} = (\boldsymbol{I}_N\otimes P^{-1})e$, one can obtain that

$$\begin{aligned}\dot{\tilde{V}}_1 = {} & \bar{e}^T(\mathcal{L}\otimes(AP + PA^T) - 2\bar{\alpha}\mathcal{L}^2\otimes BB^T)\bar{e} \\ & - 2\bar{\beta}\|(\mathcal{L}\otimes B^T)\bar{e}\|_1 + 2\bar{e}^T(\mathcal{L}\otimes B)(\epsilon + d) \\ & - 2\sum_{i=1}^{N}\mathrm{tr}(\sigma_i(\hat{W}_i - \overline{W}_i)^T(\hat{W}_i - \overline{W}_i)). \end{aligned} \tag{19}$$

Furthermore, the Hölder inequality guarantees that

$$\begin{aligned} 2\bar{e}^T(\mathcal{L}\otimes B)(\epsilon + d) & \le 2\|(\mathcal{L}\otimes B^T)\bar{e}\|_1(\|\epsilon\|_\infty + \|d\|_\infty) \\ & \le 2(\epsilon_M + d_M)\|(\mathcal{L}\otimes B^T)\bar{e}\|_1 \end{aligned} \tag{20}$$

where (8) is used to obtain the last inequality. Then, it follows from (19) and (20) that:

$$\begin{aligned}\dot{\tilde{V}}_1 \le {} & \bar{e}^T(\mathcal{L}\otimes(AP + PA^T) - 2\bar{\alpha}\mathcal{L}^2\otimes BB^T)\bar{e} \\ & - 2(\bar{\beta} - \epsilon_M - d_M)\|(\mathcal{L}\otimes B^T)\bar{e}\|_1. \end{aligned} \tag{21}$$

Since $\mathcal{G}$ is connected, it follows from Lemma 1 that $\mathcal{L}$ is positive semi-definite with a simple eigenvalue 0. Let $U = [(\mathbf{1}/\sqrt{N}), Y_1]$, with $Y_1 \in \mathbb{R}^{N\times(N-1)}$, be a unitary matrix such that $U^T\mathcal{L}U = \Lambda \triangleq \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$, where $\lambda_2 \le \cdots \le \lambda_N$ are the positive eigenvalues of $\mathcal{L}$. By further letting $\hat{e} = \mathrm{col}(\hat{e}_1, \ldots, \hat{e}_N) = (U^T\otimes\boldsymbol{I}_n)\bar{e}$, then $\hat{e}_1 = ((\mathbf{1}^T/\sqrt{N})\otimes\boldsymbol{I}_n)\bar{e} = \mathbf{0}$. By choosing $\bar{\alpha}$ and $\bar{\beta}$ sufficiently large such that $\bar{\alpha} > (\eta/2\lambda_2)$ and $\bar{\beta} > \epsilon_M + d_M$, it can be deduced from (21) and LMI (9) that

$$\begin{aligned}\dot{\tilde{V}}_1 & \le \bar{e}^T(\mathcal{L}\otimes(AP + PA^T) - 2\bar{\alpha}\mathcal{L}^2\otimes BB^T)\bar{e} \\ & = \hat{e}^T(\Lambda\otimes(AP + PA^T) - 2\bar{\alpha}\Lambda^2\otimes BB^T)\hat{e} \\ & = \sum_{i=2}^{N}\lambda_i\hat{e}_i^T(AP + PA^T - 2\bar{\alpha}\lambda_iBB^T)\hat{e}_i \\ & \le -\theta\hat{e}^T(\Lambda\otimes P)\hat{e} \\ & = -\theta e^T(\mathcal{L}\otimes P^{-1})e \end{aligned} \tag{22}$$

where θ is the positive scalar defined in LMI (9). Since $\mathcal{L} \ge 0$ and $P > 0$, if follows that $V_1(t)$ is nonincreasing, which guarantees that all the signals e, $\alpha_i(t)$, $\beta_i(t)$, $\tilde{W}_i$ and $\tilde{\overline{W}}_i$ in $V_1(t)$ are bounded. From (7) and (8), $\dot{e}$ is also bounded, then the function $\theta e^T(t)(\mathcal{L}\otimes P^{-1})e(t)$ is uniformly continuous. Since $V_1(t) \le V_1(0)$ and is nonincreasing, it thus has a finite limit V_1^∞ as $t\to\infty$. By noting (22), one can see that

$$\int_0^{+\infty}\theta e^T(t)(\mathcal{L}\otimes P^{-1})e(t)dt \le V_1(0) - V_1^\infty. \tag{23}$$

By utilizing Lemma 2, one has $\lim_{t\to\infty}\theta e^T(t)(\mathcal{L}\otimes P^{-1})e(t) = 0$. Note that $(\mathbf{1}_N^T\otimes\boldsymbol{I}_n)e = \mathbf{0}$, one has $\lim_{t\to\infty}\|e(t)\| = 0$.

On the other hand, denote $\hat{W}_i^e = \hat{W}_i - \overline{W}_i$, then it follows from (6) that:

$$\dot{\hat{W}}_i^e = -\sigma_i(\tau_i + \pi_i)\hat{W}_i^e + \tau_i S_i(x_i)\delta_i^T P^{-1} B. \tag{24}$$

Since $\lim_{t\to\infty}\|\delta(t)\| = 0$ as $\lim_{t\to\infty}\|e(t)\| = 0$, $S_i(x_i)$ is uniformly bounded, σ_i, τ_i and π_i are given positive constants, then $\lim_{t\to\infty}\|\hat{W}_i^e\|_F = 0$, i.e., $\lim_{t\to\infty}(\hat{W}_i - \overline{W}_i) = \boldsymbol{O}$. By noting from (5) that κ_i, $\nu_i > 0$ and $\Gamma > 0$, $\alpha_i(t)$ and $\beta_i(t)$ are designed to be monotonically increasing, thus, each of them converges to a finite value due to the upper boundness. The proof is completed. ■

Remark 4: LMI (9) is feasible iff (A,B) is stabilizable, where the detailed analysis can be found in [37]. Note that one can also formulate the criteria in the forms of Riccati inequality $A^T P + PA + Q - PBB^T P \leq 0$ as in [34], whose solvability for some $P > 0$ is equivalent to that of LMI (9).

Remark 5: It can be seen from the proof of Theorem 1 that the introduction of pseudo ideal weight matrices in (6) compensates the ultimately uniformly bounds of the consensus errors (see [33], [34], [36]) and paves the way to fully distributed fashion of the information.

Remark 6: When selecting constants κ_i, ν_i, τ_i, σ_i $\pi_i \in \mathbb{R}^+$, τ_i and π_i should be chosen relatively large so that $\hat{W}_i$ and $\overline{W}_i$ converge with satisfactory rates. Small constant σ_i aims to introduce the σ-modifications to improve the robustness and agility of the controllers, which can prevent the estimates $\hat{W}_i$ from shifting to very high values [45] and guarantee the quickness of the stabilization process. Finally, κ_i and ν_i determine the step size of the dynamic coupling weights and should both be small to ensure the stability of the closed-loop system.

Corollary 1: Under Assumptions 1 and 2, the asymptotic consensus problem of MAS (2) can be solved under distributed adaptive protocol

$$u_i = \alpha F\delta_i + \beta \operatorname{sgn}(F\delta_i) - \hat{W}_i^T(t)S_i(x_i) \tag{25}$$

and NN weights adaptive law (6) with $F = -B^T P^{-1}$, $\alpha > (\eta/2\lambda_2)$, $\beta > \epsilon_M + d_M$, and scalars τ_i, σ_i, $\pi_i \in \mathbb{R}^+$, where P and η are defined in (9). Moreover, the estimated NN weight matrix $\hat{W}_i$ converges to the pseudo ideal matrix $\overline{W}_i$.

Remark 7: Although it has been shown that $\lim_{t\to\infty}(\hat{W}_i - \overline{W}_i) = 0$ for each agent i in both Theorem 1 and Corollary 1, it is still not clear whether the estimated NN weight $\hat{W}_i$ converges to its corresponding ideal weight matrix W_i. Nevertheless, the pseudo ideal matrix servers as a satisfactory replacement, enabling us to drive the consensus error vector to zero asymptotically.

IV. Leader–Follower Consensus Tracking via Distributed Neuroadaptive Control

In Section III, the asymptotic leaderless consensus of the uncertain MASs has been achieved. However, the consensus value is dependent on the initial condition of the agents and generally is hard to be known prior. Note that in many realistic cases, the final consensus value is needed to be specified as a prior. Thus, in this section, we consider the MAS in the presence of reference signals, i.e., the leader agent(s).

A. Consensus Tracking of a Leader With Bounded Input

In this subsection, the proposed dynamic neuro-adaptive protocol (5) is extended to solve the consensus tracking problem of leader–follower MASs with a leader of possibly nonzero input.

Suppose there are N follower agents modeled as (2) with a leader agent $N+1$ modeled by a deterministic linear system as

$$\dot{x}_{N+1} = Ax_{N+1} + Bu_{N+1}. \tag{26}$$

The leader's input is assumed to satisfy a mild boundness condition, that is

$$\|u_{N+1}\|_\infty \leq \gamma \tag{27}$$

where $\gamma > 0$ is a positive constant which could be unknown. Then, the control objective is to design distributed controllers for the followers to realize asymptotic consensus tracking, i.e., $\lim_{t\to\infty}\|x_i(t) - x_{N+1}(t)\| = 0$, $1 \leq i \leq N$.

Similarly, we use $\bar{\mathcal{G}}(\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ to represent the graph involving the leader $N+1$, where $\bar{\mathcal{A}} = (\bar{a}_{ij})$ is the augmented adjacency matrix. Since the leader always serves as a command generator, it is reasonable to assume that the behaviors of the leader will not be affected by those of followers. Thus, the Laplacian matrix of $\bar{\mathcal{G}}$ can be partitioned as

$$\bar{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_1 & -\bar{\mathbf{a}} \\ \mathbf{0}_N^T & 0 \end{pmatrix} \tag{28}$$

where $\mathcal{L}_1 = \mathcal{L} + \operatorname{diag}(\bar{\mathbf{a}})$, $\bar{\mathbf{a}} = (\bar{a}_{1(N+1)}, \ldots, \bar{a}_{N(N+1)})^T$, and $\bar{a}_{i(N+1)} > 0$ iff follower i can directly access the states of the leader.

Assumption 3: There is at least one follower who is able to receive the information of the leader.

Under Assumptions 2 and 3, one can conclude that $\mathcal{L}_1 > 0$ by Lemma 1. Define the tracking error and the local error measurement of follower i as $\phi_i = x_i - x_{N+1}$ and $\varphi_i = \sum_{j=1}^{N+1}\bar{a}_{ij}(x_i - x_j)$, respectively. Let $x = \operatorname{col}(x_1, x_2, \ldots, x_N)$, $\phi = \operatorname{col}(\phi_1, \phi_2, \ldots, \phi_N)$ and $\varphi = \operatorname{col}(\varphi_1, \varphi_2, \ldots, \varphi_N)$, then we have $\varphi = (\mathcal{L}_1 \otimes \boldsymbol{I}_n)\phi$.

The distributed neuro-adaptive protocol (5) can be smoothly modified as follows to solve the consensus tracking problem:

$$\begin{aligned} u_i &= \alpha_i(t)F\varphi_i + \beta_i(t)\operatorname{sgn}(F\varphi_i) - \hat{W}_i^T(t)S_i(x_i) \\ \dot{\alpha}_i &= \kappa_i \varphi_i^T \Gamma \varphi_i \\ \dot{\beta}_i &= \nu_i \|F\varphi_i\|_1 \end{aligned} \tag{29}$$

where the NN adaptive law can be accordingly modified as

$$\begin{aligned} \dot{\hat{W}}_i &= \tau_i\big[S_i(x_i)\varphi_i^T P^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i(t))\big] \\ \dot{\overline{W}}_i &= \sigma_i\pi_i(\hat{W}_i - \overline{W}_i). \end{aligned} \tag{30}$$

The following theorem summarizes the above protocol for the consensus tracking task.

Theorem 2: Under Assumptions 1–3, the asymptotic consensus tracking problem of the MAS with followers (2) and the leader (26) can be solved under distributed dynamic neuro-adaptive protocol (29) along with NN weight adaptive law (30) with F, Γ given as in Theorem 1 and constants κ_i, ν_i, τ_i, σ_i $\pi_i \in \mathbb{R}^+$. Moreover, each estimated NN weight matrix

$\hat{W}_i$ converges to the corresponding pseudo ideal weight matrix $\overline{W}_i$ and each coupling gain $\alpha_i(t)$ as well as $\beta_i(t)$ converges to some finite value.

Proof: The dynamics of error system ϕ can be obtained following (2), (4), (26), and (29) as:

$$\begin{aligned}\dot{\phi} &= (\boldsymbol{I}_N \otimes A + \mathcal{M}_1\mathcal{L}_1 \otimes BF)\phi \\ &\quad + (\mathcal{M}_2 \otimes B)\mathrm{sgn}((\mathcal{L}_1 \otimes F)\phi) \\ &\quad - (\boldsymbol{I}_N \otimes B)(\tilde{W}^T S(x) + \boldsymbol{u}_{N+1} - \epsilon - d) \end{aligned} \quad (31)$$

where $\mathcal{M}_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, $\mathcal{M}_2 = \mathrm{diag}(\beta_1, \ldots, \beta_N)$. $\tilde{W}$, $S(x)$, ϵ and d are defined the same as in (7) and $\boldsymbol{u}_{N+1} = \mathbf{1}_N \otimes u_{N+1}$.

Consider the following Lyapunov function candidate:

$$\begin{aligned}V_2 &= \phi^T(\mathcal{L}_1 \otimes P^{-1})\phi \\ &\quad + \sum_{i=1}^{N} \frac{1}{\kappa_i}(\alpha_i(t) - \tilde{\alpha})^2 + \sum_{i=1}^{N} \frac{1}{\nu_i}(\beta_i(t) - \tilde{\beta})^2 \\ &\quad + \sum_{i=1}^{N} \mathrm{tr}\left(\frac{1}{\tau_i}\tilde{W}_i^T\tilde{W}_i\right) + \sum_{i=1}^{N} \mathrm{tr}\left(\frac{1}{\pi_i}\tilde{\overline{W}}_i^T\tilde{\overline{W}}_i\right) \end{aligned} \quad (32)$$

where $\tilde{\overline{W}}_i$ is defined the same as that in (10) and $\tilde{\alpha}$, $\tilde{\beta}$ are two positive constants to be determined later.

The set-valued Lie derivative of V_2 along (31) can be obtained as

$$\begin{aligned}\dot{\tilde{V}}_2 &= 2\phi^T(\mathcal{L}_1 \otimes P^{-1}A + \mathcal{L}_1\mathcal{M}_1\mathcal{L}_1 \otimes P^{-1}BF)\phi \\ &\quad + \sum_{i=1}^{N} \frac{2}{\kappa_i}(\alpha_i(t) - \tilde{\alpha})\dot{\alpha}_i + \sum_{i=1}^{N} \frac{2}{\nu_i}(\beta_i(t) - \tilde{\beta})\dot{\beta}_i \\ &\quad - 2\phi^T(\mathcal{L}_1 \otimes P^{-1}B)(\tilde{W}^T S(x) + \boldsymbol{u}_{N+1} - \epsilon - d) \\ &\quad + 2\sum_{i=1}^{N} \mathrm{tr}\left(\frac{1}{\tau_i}\tilde{W}_i^T\dot{\hat{W}}_i\right) + 2\sum_{i=1}^{N} \mathrm{tr}\left(\frac{1}{\pi_i}\tilde{\overline{W}}_i^T\dot{\overline{W}}_i\right) \\ &\quad + 2\mathcal{K}[\phi^T(\mathcal{L}_1\mathcal{M}_2 \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes F)\phi)]. \end{aligned} \quad (33)$$

Substituting $\dot{\alpha}_i$ and $\dot{\beta}_i$ in (29), $\dot{\hat{W}}_i$ and $\dot{\overline{W}}_i$ in (30) into (33) gives

$$\begin{aligned}\dot{\tilde{V}}_2 &= 2\phi^T(\mathcal{L}_1 \otimes P^{-1}A + \mathcal{L}_1\mathcal{M}_1\mathcal{L}_1 \otimes P^{-1}BF)\phi \\ &\quad + 2\sum_{i=1}^{N}(\alpha_i(t) - \tilde{\alpha})\varphi_i^T\Gamma\varphi_i + 2\sum_{i=1}^{N}(\beta_i(t) - \tilde{\beta})\|F\varphi_i\|_1 \\ &\quad - 2\phi^T(\mathcal{L}_1 \otimes P^{-1}B)(\tilde{W}^T S(x) + \boldsymbol{u}_{N+1} - \epsilon - d) \\ &\quad + 2\sum_{i=1}^{N} \mathrm{tr}\big(\tilde{W}_i^T(S_i(x_i)\varphi_i^T P^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i))\big) \\ &\quad + 2\sum_{i=1}^{N} \mathrm{tr}\big(\sigma_i\tilde{\overline{W}}_i^T(\hat{W}_i - \overline{W}_i)\big) \\ &\quad + 2\mathcal{K}[\phi^T(\mathcal{L}_1\mathcal{M}_2 \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes F)\phi)]. \end{aligned} \quad (34)$$

Since $F = -B^T P^{-1}$ and $\Gamma = P^{-1}BB^T P^{-1}$, some mathematical manipulations give that

$$\phi^T(\mathcal{L}_1\mathcal{M}_1\mathcal{L}_1 \otimes P^{-1}BF)\phi = -\sum_{i=1}^{N}\alpha_i(t)\varphi_i^T\Gamma\varphi_i \quad (35)$$

and

$$\phi^T(\mathcal{L}_1\mathcal{M}_2 \otimes P^{-1}B)\mathrm{sgn}((\mathcal{L}_1 \otimes F)\phi) = -\sum_{i=1}^{N}\beta_i(t)\|F\varphi_i\|_1. \quad (36)$$

By further noting that

$$\phi^T(\mathcal{L}_1 \otimes P^{-1}B)(\tilde{W}^T S(x)) = \sum_{i=1}^{N}\mathrm{tr}\big(\tilde{W}_i^T S_i(x_i)\varphi_i^T P^{-1}B\big) \quad (37)$$

(34) can be simplified as

$$\begin{aligned}\dot{\tilde{V}}_2 &= 2\phi^T(\mathcal{L}_1 \otimes P^{-1}A)\phi \\ &\quad - 2\tilde{\alpha}\sum_{i=1}^{N}\varphi_i^T\Gamma\varphi_i - 2\tilde{\beta}\sum_{i=1}^{N}\|F\varphi_i\|_1 \\ &\quad - 2e^T(\mathcal{L}_1 \otimes P^{-1}B)(\boldsymbol{u}_{N+1} - \epsilon - d) \\ &\quad - 2\sum_{i=1}^{N}\mathrm{tr}(\sigma_i(\hat{W}_i - \overline{W}_i)^T(\hat{W}_i - \overline{W}_i)). \end{aligned} \quad (38)$$

Let $\bar{\phi} = (\boldsymbol{I}_N \otimes P^{-1})\phi$, one can obtain that

$$\begin{aligned}\dot{\tilde{V}}_2 &\leq \bar{\phi}^T(\mathcal{L}_1 \otimes (AP + PA^T) - 2\tilde{\alpha}\mathcal{L}_1^2 \otimes BB^T)\bar{\phi} \\ &\quad - 2\tilde{\beta}\|(\mathcal{L}_1 \otimes B^T)\bar{\phi}\|_1 \\ &\quad - 2\bar{\phi}^T(\mathcal{L}_1 \otimes B)(\boldsymbol{u}_{N+1} - \epsilon - d). \end{aligned} \quad (39)$$

Moreover, the Hölder inequality guarantees that

$$\begin{aligned}&-2\bar{\phi}^T(\mathcal{L}_1 \otimes B)(\boldsymbol{u}_{N+1} - \epsilon - d) \\ &\quad \leq 2\|(\mathcal{L}_1 \otimes B^T)\bar{\phi}\|_1\|\boldsymbol{u}_{N+1} - \epsilon - d\|_\infty \\ &\quad \leq 2(\gamma + \epsilon_M + d_M)\|(\mathcal{L}_1 \otimes B^T)\bar{\phi}\|_1 \end{aligned} \quad (40)$$

where (8) and (27) are used to obtain the last inequality. Then, it follows from (39) and (40) that:

$$\begin{aligned}\dot{\tilde{V}}_2 &\leq \bar{\phi}^T(\mathcal{L}_1 \otimes (AP + PA^T) - 2\tilde{\alpha}\mathcal{L}_1^2 \otimes BB^T)\bar{\phi} \\ &\quad - 2(\tilde{\beta} - \gamma - \epsilon_M - d_M)\|(\mathcal{L}_1 \otimes B^T)\bar{\phi}\|_1. \end{aligned} \quad (41)$$

Let $\Delta = \mathrm{diag}(\Delta_i)$ be the diagonal matrix associated with $\mathcal{L}_1$, that is, there exists a unitary matrix R such that $R^T\mathcal{L}_1 R = \Delta$. Let $\hat{\phi} = (R^T \otimes \boldsymbol{I})\bar{\phi}$, then by choosing $\tilde{\alpha}$ and $\tilde{\beta}$ sufficiently large such that $\tilde{\alpha} > (\eta/(2\lambda_{\min}(\mathcal{L}_1)))$ and $\tilde{\beta} > \gamma + \epsilon_M + d_M$, one has

$$\begin{aligned}\dot{\tilde{V}}_2 &\leq \bar{\phi}^T\big(\mathcal{L}_1 \otimes (AP + PA^T) - 2\tilde{\alpha}\mathcal{L}_1^2 \otimes BB^T\big)\bar{\phi} \\ &= \sum_{i=1}^{N}\Delta_i\hat{\phi}_i^T(AP + PA^T - 2\tilde{\alpha}\Delta_i BB^T)\hat{\phi}_i \\ &\leq -\theta\phi^T(\mathcal{L}_1 \otimes P^{-1})\phi \end{aligned} \quad (42)$$

where θ is a positive scalar defined in LMI (9). Since $\mathcal{L}_1 > 0$ and $P > 0$, it follows that $V_2(t)$ is nonincreasing, which guarantees that all the signals ϕ, $\alpha_i(t)$, $\beta_i(t)$, $\tilde{W}_i$, and $\tilde{\overline{W}}_i$ in $V_2(t)$ are bounded. From (31), $\dot{\phi}$ is also bounded. Similar to the proof of theorem 1, $\lim_{t\to\infty}\|\phi(t)\| = 0$, $\lim_{t\to\infty}(\hat{W}_i - \overline{W}_i) = \boldsymbol{O}$, and each of the coupling weights $\alpha_i(t)$ and $\beta_i(t)$ converges to a finite value due to the upper boundness. This completes the proof. ■

Corollary 2: Under Assumptions 1–3, the asymptotic consensus tracking of the MAS with followers (2) and the leader (26) can be solved under static distributed neuroadaptive protocol

$$u_i = \alpha F\varphi_i + \beta \operatorname{sgn}(F\varphi_i) - \hat{W}_i^T(t)S_i(x_i)$$

along with NN adaptive law (30) with $F = -B^T P^{-1}$, $\alpha > (\eta/(2\lambda_{\min}(\mathcal{L}_1)))$, $\beta > \gamma + \epsilon_M + d_M$, and scalars τ_i, σ_i $\pi_i \in \mathbb{R}^+$, where P and η are defined in (9). Moreover, each estimated NN weight matrix $\hat{W}_i$ converges to the corresponding pseudoideal matrix $\overline{W}_i$.

B. Extensions to Containment Protocol With Multiple Leaders

In this subsection, the results Section IV-A are extended to the MAS with multiple leaders. Suppose that there are M leaders in the MAS described as

$$\dot{x}_{N+l} = Ax_{N+l} + Bu_{N+l} \tag{43}$$

where the external control inputs are uniformly bound by

$$\|u_{N+l}\|_\infty \leq \hat{\gamma}. \quad l = 1, \ldots, M. \tag{44}$$

Then, the control objective is to solve the asymptotic containment problem defined as follows.

Definition 1: The followers (2) are said to realize asymptotic containment by multiple leaders (43), if there exist some nonnegative scalars $q_{ij} \geq 0$ with $\sum_{j=1}^M q_{ij} = 1$, such that

$$\lim_{t\to\infty} \|x_i(t) - \sum_{j=1}^M q_{ij} x_j(t)\| = 0, \quad i = 1, \ldots, N.$$

Let $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ represent the graph involving the leaders, where $\hat{\mathcal{A}} = (\hat{a}_{ij})$ is the augmented adjacency matrix. Then, the Laplacian matrix of $\hat{\mathcal{G}}$ can be partitioned as

$$\hat{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_2 & -\hat{\boldsymbol{a}} \\ \boldsymbol{O}_{M\times N} & \boldsymbol{O}_M \end{pmatrix} \tag{45}$$

where $\mathcal{L}_2 = \mathcal{L} + \sum_{l=1}^M \operatorname{diag}(\hat{\mathbf{a}}_l)$, $\hat{\boldsymbol{a}} = (\hat{\mathbf{a}}_1, \ldots, \hat{\mathbf{a}}_M) \in \mathbb{R}_{N\times M}$, and $\hat{a}_{il} > 0$ iff follower i can directly obtain the states of the leader l.

Assumption 4: For each leader, there exists at least one follower with access to its information.

Under Assumptions 2 and 4, one can conclude that $\mathcal{L}_2 > 0$ and $\hat{\boldsymbol{a}}\mathbf{1} = \mathcal{L}_2\mathbf{1}$. If $Q = (q_{ij}) = \mathcal{L}_2^{-1}\hat{\boldsymbol{a}}$, then $q_{ij} \geq 0$ and $\sum_{j=1}^M q_{ij} = 1$, which makes Q a qualified candidate for the matrix of containment coefficients. Denote $\rho_i = x_i - \sum_{j=1}^M q_{ij}x_j$ as the containment error and $\zeta_i = \sum_{j=1}^{N+M} \hat{a}_{ij}(x_i - x_j)$ as the local error measurement for follower i, respectively. In a compact form, $\rho = x - (\mathcal{L}_2^{-1}\hat{\boldsymbol{a}} \otimes \boldsymbol{I}_n)x_L$, $\zeta = (\mathcal{L}_2 \otimes \boldsymbol{I}_n)x - (\hat{\boldsymbol{a}} \otimes \boldsymbol{I}_n)x_L$, where $\rho = \operatorname{col}(\rho_1, \ldots, \rho_N)$, $\zeta = \operatorname{col}(\zeta_1, \ldots, \zeta_N)$, $x = \operatorname{col}(x_1, \ldots, x_N)$ and $x_L = \operatorname{col}(x_{N+1}, \ldots, x_{N+M})$. Then, it is obvious that $\zeta = (\mathcal{L}_2 \otimes \boldsymbol{I}_n)\rho$.

Similarly, the distributed neuroadaptive protocol for the followers can be designed as:

$$\begin{aligned} u_i &= \alpha_i(t)F\zeta_i + \beta_i(t)\operatorname{sgn}(F\zeta_i) - \hat{W}_i^T(t)S_i(x_i) \\ \dot{\alpha}_i &= \kappa_i\zeta_i^T\Gamma\zeta_i \\ \dot{\beta}_i &= \nu_i\|F\zeta_i\|_1 \end{aligned} \tag{46}$$

with NN adaptive law

$$\begin{aligned} \dot{\hat{W}}_i &= \tau_i\big[S_i(x_i)\zeta_i^T P^{-1}B - \sigma_i(\hat{W}_i - \overline{W}_i(t))\big] \\ \dot{\overline{W}}_i &= \sigma_i\pi_i(\hat{W}_i - \overline{W}_i). \end{aligned} \tag{47}$$

Theorem 3: Under Assumptions 1, 2, and 4, the asymptotic containment of MAS with followers (2) and multiple leaders (43) can be solved under distributed dynamic neuroadaptive protocol (46) along with NN adaptive law (47) with F and Γ given as in Theorem 1 and constants κ_i, ν_i, τ_i, σ_i $\pi_i \in \mathbb{R}^+$. Moreover, each estimated NN weight matrix $\hat{W}_i$ converges to the corresponding pseudo ideal weight matrix $\overline{W}_i$ and each coupling gain $\alpha_i(t)$ as well as $\beta_i(t)$ converges to some finite value.

Proof: The dynamics of error system ρ can be obtained following (2), (4), (43), and (46) as:

$$\begin{aligned} \dot{\rho} &= (\boldsymbol{I}_N \otimes A + \mathcal{M}_1\mathcal{L}_2 \otimes BF)\rho \\ &\quad + (\mathcal{M}_2 \otimes B)\operatorname{sgn}((\mathcal{L}_2 \otimes F)\rho) \\ &\quad - (\boldsymbol{I}_N \otimes B)(\tilde{W}^T S(x) - \epsilon - d) - \big(\mathcal{L}_2^{-1}\hat{\boldsymbol{a}} \otimes B\big)\boldsymbol{u}_L \end{aligned} \tag{48}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are defined as before without ambiguity and $\boldsymbol{u}_L = \operatorname{col}(u_{N+1}, \ldots, u_{N+M})$.

The following proof is similar to that of Theorem 2 when considering Lyapunov function candidate:

$$\begin{aligned} V_3 &= \rho^T(\mathcal{L}_2 \otimes P^{-1})\rho \\ &\quad + \sum_{i=1}^N \frac{1}{\kappa_i}(\alpha_i(t) - \hat{\alpha})^2 + \sum_{i=1}^N \frac{1}{\nu_i}(\beta_i(t) - \hat{\beta})^2 \\ &\quad + \sum_{i=1}^N \operatorname{tr}\left(\frac{1}{\tau_i}\tilde{W}_i^T\tilde{W}_i\right) + \sum_{i=1}^N \operatorname{tr}\left(\frac{1}{\pi_i}\tilde{\overline{W}}_i^T\tilde{\overline{W}}_i\right) \end{aligned} \tag{49}$$

and thus be omitted due to the space limitation. ■

Corollary 3: Under Assumptions 1, 2, and 4, the asymptotic containment of MAS with followers (2) and multiple leaders (43) can be solved under static distributed neuro-adaptive protocol:

$$u_i = \alpha F\zeta_i + \beta\operatorname{sgn}(F\zeta_i) - \hat{W}_i^T(t)S_i(x_i)$$

along with NN adaptive law (47) with $F = -B^T P^{-1}$, $\alpha > (\eta/(2\lambda_{\min}(\mathcal{L}_2)))$, $\beta > \hat{\gamma} + \epsilon_M + d_M$, and scalars τ_i, σ_i $\pi_i \in \mathbb{R}^+$, where P and η are defined in (9). Moreover, each estimated NN weight matrix $\hat{W}_i$ converges to the corresponding pseudoideal weight matrix $\overline{W}_i$.

Remark 8: When comparing the previous theorems and their corresponding corollaries, the incorporation of adaptive couplings is obviously necessary to accomplish fully distributed schemes without any global information. Similar techniques can also be found in [28] and [43]. It should also be pointed out that another powerful tool to implement fully distributed control for uncertain network systems is the adaptive backstepping technique (see [32], [46] for details).

Remark 9: Although it is theoretically proved in Theorem 3 that the containment errors converge to zero, a practical issue when implementing the dynamic protocol (46) is that the coupling gains $\alpha_i(t)$ and $\beta_i(t)$ might always increase slowly due to measurement errors, chattering phenomenon, or external disturbances [20]. A practical solution to this issue is stated

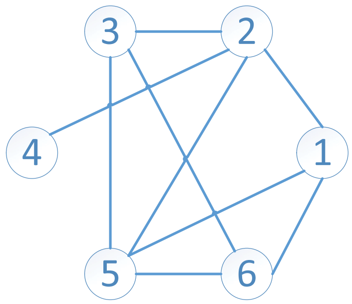

Fig. 1. Communication topology for Example 1.

in the following. Introduce a small constant $r_i > 0$ for follower i and update $\alpha_i(t)$ and $\beta_i(t)$ following (46) whenever $\|\zeta_i\|_2 > r_i$; otherwise, let $\dot{\alpha}_i(t) = \dot{\beta}_i(t) = 0$. In this way, as long as the tracking error of each follower converges into a desirable bound, the coupling strengths will converge to some finite values. Note that the above discussions are also applicable to the dynamic consensus protocol (5) and dynamic tracking protocol (29).

V. Numerical Examples

In this section, three numerical examples are provided to validate the obtained theoretical results.

Example 1 (Leaderless Consensus): Consider an MAS with six agents, the communication graph among which is shown in Fig. 1. The linear part dynamics of the agents is described by the linearized model of the longitudinal dynamics of an aircraft [47], which follows (2) with $x_i = (x_{i1}, x_{i2}, x_{i3})^T$:

$$A = \begin{pmatrix} -0.277 & 1 & -0.0002 \\ -17.1 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 6.67 \end{pmatrix}$$

in which x_{i1}, x_{i2}, and x_{i3} represent the attacking angle, pitch state, and elevator angle, respectively. The unknown nonlinearities are chosen as $f_i(x_i(t)) = 2x_{i1}(t)\sin(i + x_{i1}(t)) + (x_{i2}(t))^2\cos(x_{i3}(t))$. Disturbances $d_i(t) = 0.1\sin(it)$, which is bounded by $d_M = 0.1$. To illustrate Theorem 1, let $\eta = \theta = 2$, solve LMI (9), and obtain

$$F = (1.9236, 1.4825, -5.2431)$$

and

$$\Gamma = \begin{pmatrix} 3.7004 & 2.8518 & -10.0858 \\ 2.8518 & 2.1978 & -7.7729 \\ -10.0858 & -7.7729 & 27.4899 \end{pmatrix}.$$

In each NN approximator, six hidden neurons with sigmoid activation functions are used to approximate the unknown nonlinearities and the input layer weight matrix is chosen randomly from normal distribution. Select $\kappa_i = \nu_i = 0.01$ in (5), $\tau_i = 210$, $\sigma_i = 0.5$, and $\pi_i = 5$ in (6) and initialize $\hat{W}_i$ and $\overline{W}_i$ as zero matrices. When implementing (5), we limit the consensus error bound for each agent to be 0.05 considering Remark 9. Finally, the state trajectories of the agents are shown in Fig. 2. The elementwise values of pseudoconverge errors $\hat{W}_i^e$ are shown in Fig. 3, and the coupling weights $\alpha_i(t)$ and $\beta_i(t)$ are shown in Fig. 4. It can be seen that all the coupling strengths remain unchanged after about $t = 4$.

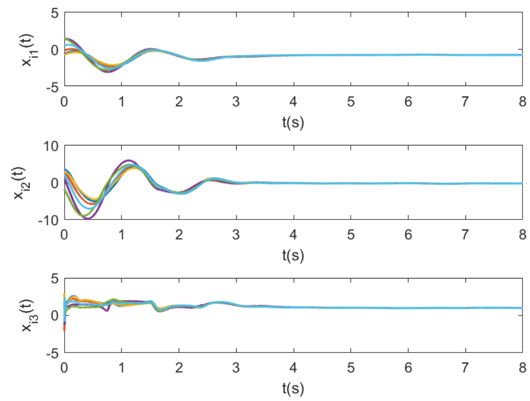

Fig. 2. States of agents $x_i(t)$, $i = 1, \ldots, 6$, in Example 1.

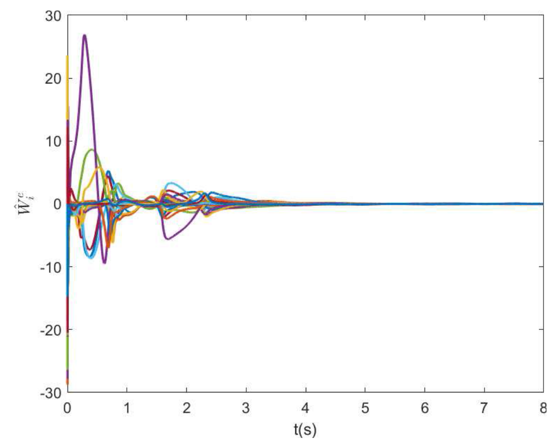

Fig. 3. Elementwise values of $\hat{W}_i^e$, $i = 1, \ldots, 6$, in Example 1.

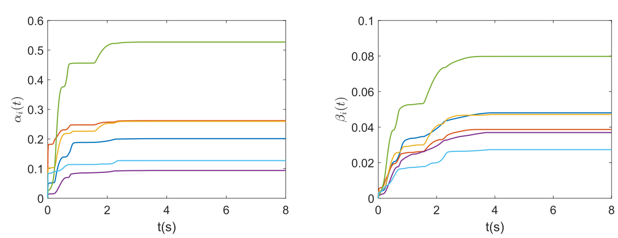

Fig. 4. Coupling weights $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, \ldots, 6$, in Example 1.

Example 2 (Tracking of a Leader With Bounded Input): Consider a leader–follower MAS with six followers, whose dynamics are given in (2) with:

$$x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$f_i(x_i(t)) = 2x_{i1}(t)\cos(i + x_{i2}(t))$ and $d_i(t) = 0.01\cos(it)$. The dynamics of leader 7 is governed by (26) with control input $u_7 = K_L x_7$ with feedback gain matrix $K_L = (0, -2)$. Then, it can be easily verified that the closed-loop dynamics of the leader is an oscillator. In this case, the feedback controller u_7 is bounded by γ as in (27). However, the value of γ, which depends on the initial value $x_7(0)$, is probably unknown to

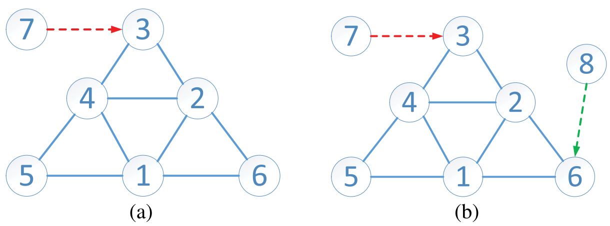

Fig. 5. Communication topologies for Example 2 (a) and Example 3 (b).

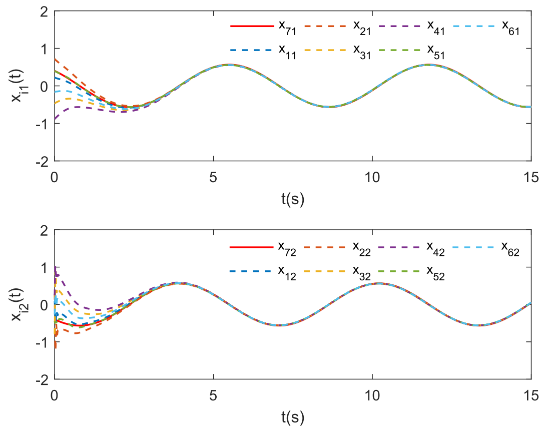

Fig. 6. States of agents $x_i(t)$, $i = 1, \ldots, 7$, in Example 2.

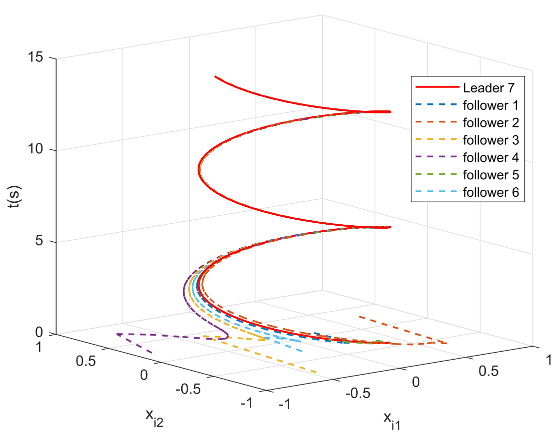

Fig. 7. Trajectories of agents $x_i(t)$, $i = 1, \ldots, 7$, in Example 2.

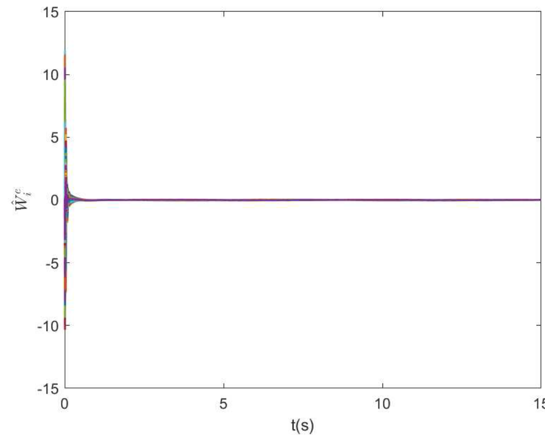

Fig. 8. Elementwise values of $\hat{W}_i^e$, $i = 1, \ldots, 6$, in Example 2.

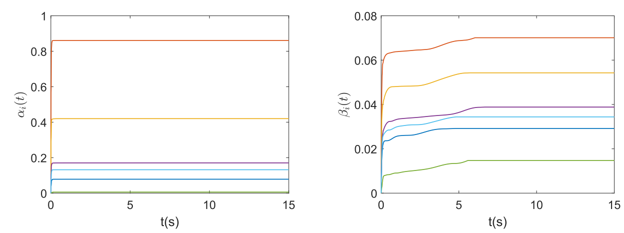

Fig. 9. Coupling weights $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, \ldots, 6$, in Example 2.

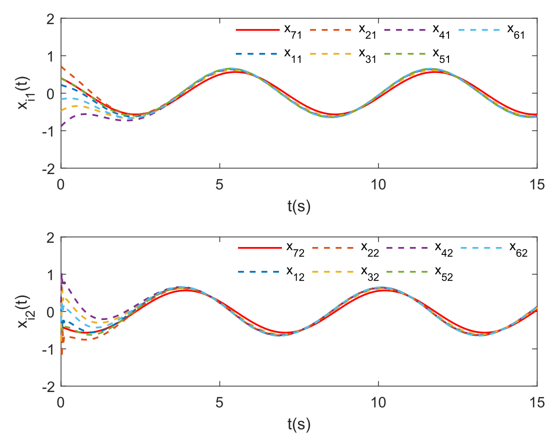

Fig. 10. States of agents $x_i(t)$, $i = 1, \ldots, 7$, without introducing $\overline{W}_i$ for followers in Example 2.

the followers. The communication graph among the agents is shown in Fig. 5(a).

To verify Theorem 2, let $\eta = 2$, $\theta = 1$, and solve LMI (9). Then, it follows that:

$$F = -(5.1621, 5.0202), \quad \Gamma = \begin{pmatrix} 26.6477 & 25.9150 \\ 25.9150 & 25.2024 \end{pmatrix}.$$

Select $\kappa_i = \nu_i = 0.1$ in (29) and $\tau_i = 100$, $\sigma_i = 0.5$, and $\pi_i = 10$ in (30). Considering Remark 9, let $r_i = 0.005$. In the simulation, the initial state of the leader agent is chosen as $x_7(0) = (0.4, -0.4)^T$ and 24 hidden neurons are assigned for each follower agent to guarantee the approximation performance. After the simulation, the state trajectories of all agents are shown in Fig. 6 and the tracking behaviors are shown in Fig. 7. The profiles of pseudoconvergence errors $\hat{W}_i^e$ are shown in Fig. 8, and Fig. 9 shows the coupling gains $\alpha_i(t)$ and $\beta_i(t)$.

Moreover, it should be noted that the steady-state values of coupling strengthes $\alpha_i(t)$ and $\beta_i(t)$ are usually smaller than those when static distributed neuroadaptive protocol in Corollary 2 (which is also the philosophy of [34] and [37]) is adopted. For example, the mean value of steady-state values of $\alpha_i(t)$ in the above is $\mathrm{mean}(0.0784, 0.8608, 0.4202, 0.1706, 0.0059, 0.1320) = 0.2780$, whereas the threshold value of α here instructed by Corollary 2 is $\lceil \eta/(2\lambda_{\min}(\mathcal{L}_1)) \rceil = 8.6391$, which indicates that the proposed dynamic controller is clearly more economical from the energy point of view.

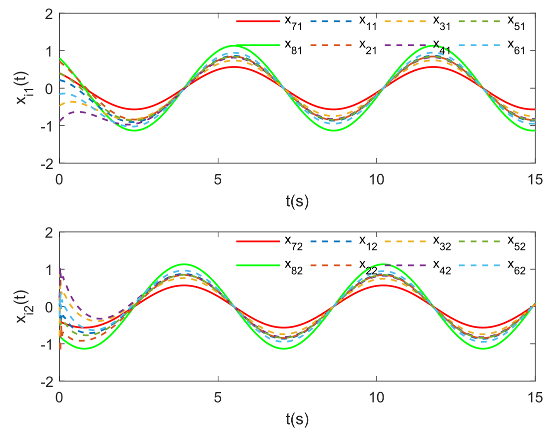

Fig. 11. States of agents $x_i(t)$, $i = 1, \ldots, 8$, in Example 3.

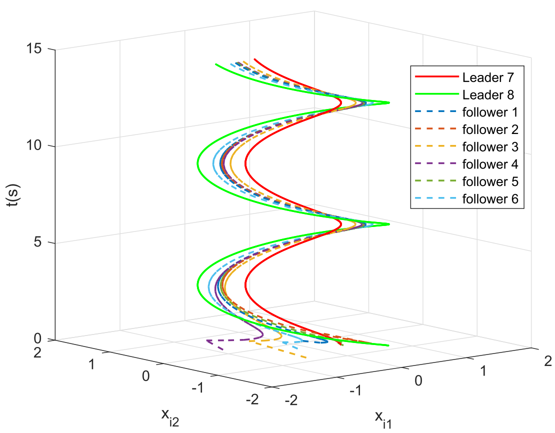

Fig. 12. Trajectories of agents $x_i(t)$, $i = 1, \ldots, 8$, in Example 3.

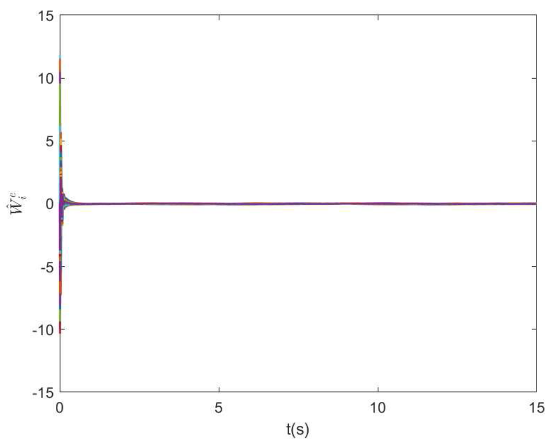

Fig. 13. Elementwise values of $\hat{W}_i^e$, $i = 1, \ldots, 6$, in Example 3.

To further highlight the asymptotic convergence property of the tracking control, the above example is also implemented without introducing pseudoideal weight matrices $\overline{W}_i$ but with σ_i-modification term $-\sigma_i \hat{W}_i$ in (30), which is the classical NN adaptive method widely used in the existing literature [33]–[36]. Keeping all the other design and parameters unchanged, the states of all agents are solved and shown

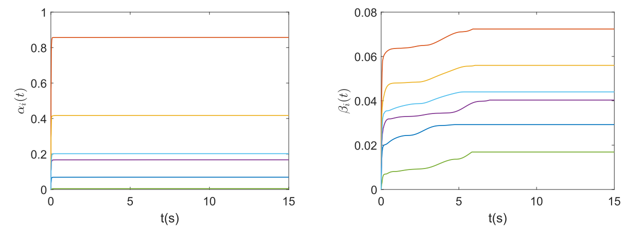

Fig. 14. Coupling weights $\alpha_i(t)$ and $\beta_i(t)$, $i = 1, \ldots, 6$, in Example 3.

in Fig. 10, where the followers track the leader with visible UUB tracking errors.

Example 3 (Containment by Multiple Leaders): Consider the MAS in Example 2 with one more leader agent 8. The overall communication topology is drawn in Fig. 5(b). The dynamics of leader 8 is the same as that of leader 7, but with different initial positions $x_8(0) = (0.8, -0.8)^T$. With protocol (46) and exactly the same design parameters as in Example 2, the asymptotic containment of the MAS can be numerically achieved. Concretely, the state trajectories of all agents are shown in Figs. 11 and 12, through $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The profiles of $\hat{W}_i^e$ are shown in Fig. 13, and Fig. 14 shows the coupling gains $\alpha_i(t)$ and $\beta_i(t)$.

It can be seen that the numerical examples verify the theoretical results very well.

VI. Conclusion

In this article, a class of uncertain MASs subject to unmodeled nonlinearities and unknown disturbances was revisited. First, with the help of nonsmooth analysis and adaptive control theory, a novel class of fully distributed neuroadaptive controller has been designed for the leaderless and asymptotic consensus by combining the ideas of neuroadaptive control and node-based dynamic coupling designs. Then, the controller has been found capable, after some natural modifications, to handle the cases in the presence of leader agents with bounded inputs, including the scenarios of tracking of a single leader and containment by multiple leaders. Finally, three numerical examples on consensus, tracking, and containment, respectively, have been performed to testify the theoretical results. Note that the topology of this article is limited to undirected graphs, which paves the way to further extensions of the control strategy for directed graphs. Also, the finite- and fixed-time schemes along this line are interesting topics.

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