Distributed Aperiodic Time-Triggered and Event-Triggered Consensus: A Scalability Viewpoint

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***Abstract*—We revisit distributed sampled-data consensus problems from a scalability point of view. Existing solutions in the literature for estimating the maximum sampling interval that preserves stability rely on the Lyapunov functional method. With this method, the overall closed-loop system (i.e. the overall network of agents) is treated as a time-delayed system. Here, a critical point is the scalability of the resulting stability conditions: in fact, the size of the LMIs to be solved depends on the size of the network. In contrast with this method, an easy-to-use and scalable method is presented, with stability conditions that are independent on the size of the network. It is shown that the proposed method can handle linear and Lipschitz nonlinear multiagent systems with both aperiodic time-triggered and event-triggered control in a unified way. Numerical examples show the efficiency of the proposed approach and the tightness of the estimated maximum sampling interval.**

***Index Terms*—Distributed control, consensus, sampled-data control, Lipschitz nonlinear multiagent systems.**

I. Introduction

SAMPLED-data control is an active research topic in the digital era. In the field of multiagent systems (MASs), sampled-data control plays an important role in consensus since, instead of assuming continuous communication among the agents, communication at discrete time instants can be allowed. For the stability analysis of sampled-data systems, a representative method is the input-delay approach, which was initially proposed for single systems [1], [2], and then extended to MASs [3], [4], [5], [6], [7], [8], [9]. The input-delay method uses the Lyapunov functional analysis, where the maximum sampling interval is treated as a free parameter so as to solve appropriate control gains. In other words, a direct estimate of the maximum sampling interval is missing. Another notable issue is that the size of the linear matrix inequalities (LMIs) sufficient conditions usually depends linearly/polynomially on the size N of the network, e.g., $\mathcal{O}(N)$ in [4], [9] and $\mathcal{O}(N^2)$ in [8]. This is because the closed-loop under consideration to apply the Lyapunov functional analysis is the whole network of agents. The dependence of the resulting LMI conditions on the number of agents in the network makes the approach not scalable. In other words, this dependence limits the applications of the input-delay approach to large network systems. Similar scalability issue also arises in sampled-data control methods based on hybrid system theory, e.g., [10], [11], [12] (the size of the LMIs is $\mathcal{O}(N)$).

At the same time, a decomposition method was proposed for first-order [13] and second-order MASs [14], and was recently applied to higher-order linear MASs [15], [16]. The main idea of the decomposition method is to transform the stability of the whole closed-loop system into the stabilization of some subsystems. As a result, a direct estimate of the maximum sampling interval is available a priori in an algebraic form. A byproduct is that the size of the LMIs to be solved is independent of the size of the network [15], [16], thus alleviating the computation burden and improving scalability. Nevertheless, such decomposition-based analysis is not yet mature, e.g., whether this framework can be adapted to nonlinear MASs is not clear, and whether this framework can cover both aperiodic time-triggered and event-triggered consensus control is also not clear.

As a branch stemming from sampled-data control, periodic event-triggered control has become very popular in recent years [17], [18], [19]. The key feature of periodic event-triggered is that the data transmission and event monitoring are only needed to be accomplished at (periodic) discrete sampling instants, in contrast with traditional continuous-time event-triggered mechanisms [20], [21]. This feature benefits control systems in at least two aspects, i.e., saving communication costs (which is in line with sampled-data control) and excluding Zeno behavior naturally (since the inter-event time is no less than the sampling period). In the field of MASs, several periodic event-triggered consensus solutions have been proposed

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for linear MASs [22], [23] and Lipschitz nonlinear MASs [24] based on the Lyapunov functional analysis, which is not scalable as discussed before: in fact, the size of the LMI conditions in [22], [23], [24] are of $\mathcal{O}(N)$. Differently, the work in [25] can be seen as a generalization of the decomposition idea in [15] from sampled-data control to periodic event-triggered control, while the results are limited to linear MASs. Besides, although periodic sampling may be satisfactory in some cases, it is usually required to use aperiodic sampling patterns to meet different operation conditions or for economical reasons [8]. For instance, smaller sampling period can be chosen at the beginning for better transient behavior, and larger sampling period can be chosen at the steady consensus state to save more energy. The solution for aperiodic sampling consensus in the literature is usually based on some refined Lyapunov functional [8], [26], [27], which is again not scalable. Moreover, designing aperiodic event-triggered consensus along this line has not been reported yet, and remains an open problem.

Motivated by the above discussions, this paper addresses distributed sampled-data leaderless consensus of linear and Lipschitz nonlinear MASs from a scalability point of view. The main contributions of this paper are summarized as follows:

1) We present a unified design for distributed aperiodic sampled-data consensus, and its extensions to the event-triggered setting by consistently generalizing the decomposition idea [13], [14], [15], [16]. To distinguish, the former is referred to as *distributed aperiodic time-triggered (DATT)* consensus, and the latter is referred to as *distributed aperiodic event-triggered (DAET)* consensus.
2) For both consensus protocols, an upper bound of the maximum sampling interval as well as a lower bound of the coupling gain between the agents are explicitly given, providing an insight on the interplay between agent-level stabilization and network-level stability.
3) Differently from most existing literature (e.g., [4], [8], [9], [10], [11], [12], [22], [23], [24], [26], [27]), the size of the LMI condition for both protocols is independent of the size N of the network, but is of $\mathcal{O}(n)$ where n is the dimension of a single agent, thus is scalable and easy-to-use.
4) As compared to [13], [14], [15], [16], the proposed method is applicable not only to linear agents but also to Lipschitz nonlinear ones. Besides, a novel DAET consensus control is developed, again applicable to both linear and Lipschitz nonlinear cases.

Notations: Denote $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}^n$ as the sets of natural numbers, positive real scalars, n-dimensional real vectors, respectively. Let $\|\cdot\|$ be the 2-norm (resp. induced 2-norm) of a vector (resp. a matrix), and $\|\cdot\|_F$ be the Frobenius norm of a matrix. If A is p-squared, denote $\lambda_1(A)$ and $\lambda_p(A)$ as its minimum and maximum singular value respectively, i.e., $\lambda_p(A) = \|A\|$ in this case. If A is further positive semi-definite, let $\lambda_2(A)$ be its minimum nonzero eigenvalue. The matrix inequality $A > 0$ means that A is positive definite. The operator $\mathrm{col}(\cdot,\cdot)$ denotes the column vectorization, and $\otimes$ is the Kronecker product. A signal $s(t) \in \mathbf{L}^\infty$ if it is essentially bounded, i.e., $\mathrm{ess}\ \sup_{t\geq 0}|s(t)| < \infty$, where "ess sup" denotes essential supremum.

II. Preliminaries and Problem Formulation

An *undirected graph* (or simply *graph*) $\mathcal{G}$ consists of an agent set $\mathcal{V}$ and an edge set $\mathcal{E}$. Square matrices associated with $\mathcal{G}$ are introduced as follows. The adjacency matrix $\mathcal{A}$ is such that $\mathcal{A}_{ij} = 1$ if there is an edge between agent i and agent j, and $\mathcal{A}_{ij} = 0$ otherwise. The degree matrix $\mathcal{D}$ is diagonal with the i-th diagonal element $\mathcal{D}_{ii}$ being the cardinality of the neighbors of i. The Laplacian matrix $\mathcal{L}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Self-loops are excluded. The graph is connected if there is a path between any pair of agents [28].

Consider N agents interacting over a graph $\mathcal{G}$. The relative error signal that can be accessed by agent i is defined as $\delta_i(t) \triangleq \sum_{j=1}^N \mathcal{A}_{ij}(x_i(t) - x_j(t))$. It is well know that the signal δ_i can be used for feedback to achieve distributed consensus [28], [29], [30]: traditionally, such signal is assumed to be acquired via continuous communications between agent i and its neighbors. As continuous communication is unpractical, in this work we will consider more practical alternatives. We first consider the sampled sequence of $\delta_i(t)$:

$$\delta_i(t_s) \triangleq \sum_{j=1}^N \mathcal{A}_{ij}(x_i(t_s) - x_j(t_s)), \tag{1}$$

where $\{t_s \in [0,\infty) | s \in \mathbb{N}, t_0 = 0\}$ is the sampling time sequence of the communication. Let $T_s \triangleq t_{s+1} - t_s \geq h$ for some $h \in \mathbb{R}_+$ be the rest time. Note that the sampling is synchronous among the agents, while not necessarily periodic [16], i.e., the sampling belongs to the class of synchronous nonuniform sampling (SNS) according to [31]. Since the samples depend on the rest time T_s, we refer to a consensus design that uses $\delta_i(t_s)$ in (1) for feedback as *Distributed Aperiodic Time-Triggered (DATT)* consensus.

As a step further, consider involving an extra 'rest' mechanism to save even more costs for communication and control updates, on top of (1). To this purpose, let us define

$$\hat{\delta}_i(t_s) \triangleq \sum_{j=1}^N \mathcal{A}_{ij}(\hat{x}_i(t_s) - \hat{x}_j(t_s)), \tag{2}$$

where $\hat{x}_i(t_s)$ is the latest broadcasted sampled state of agent i, and $\hat{x}_j(t_s)$ is the latest received sampled state from neighbor j. For each $i \in \mathcal{V}$ and each sampling instant t_s, agent i evaluates an event E_i, and broadcasts its current sampled state $x_i(t_s)$ to its neighbors (and its own controller) immediately only if E_i is triggered. In other words, $\hat{x}_i(t_s) = x_i(t_s)$ if E_i is triggered, and $\hat{x}_i(t_s) = \hat{x}_i(t_{s-1})$ otherwise. Naturally, we refer to a control design that uses $\hat{\delta}_i(t_s)$ in (2) for feedback as *Distributed Aperiodic Event-Triggered (DAET)* consensus. Note that the triggered time sequence in DAET constitutes a subsequence of the sampling time sequence.

Remark 1: It should be highlighted, in the first place, that DAET control is intrinsically Zeno-free: this is because the minimum inter-event time corresponds to the minimum sampling interval, which is already no less than some $h \in \mathbb{R}_+$. Introducing aperiodicity in sampled-data consensus is not new [8], [26], [27]. However, sufficient conditions provided in the literature do not satisfy the scalability requirements. A recent work [16] has considered DATT control with scalable estimation of the maximum sampling interval, but the results only apply to special classes of linear MASs. Besides, the authors are not aware of any DAET approach along this line.

In this paper, we aim to solve the following problems:

Q1: Design a DATT consensus controller, and estimate the maximum sampling interval in a scalable fashion, i.e., provide sufficient conditions whose size does not depend on the number of agents N in the network.

Q2: Design a DAET consensus controller and estimate the maximum sampling interval in a scalable fashion.

To close this section, we introduce an assumption followed by two technical lemmas.

Assumption 1: The communication graph $\mathcal{G}$ is connected.

Lemma 1 ([28]): Under Assumption 1, the Laplacian matrix $\mathcal{L}$ has an eigenvalue 0 with $\mathbf{1}_N$ as a corresponding eigenvector, and all the other eigenvalues have positive real parts. Moreover, $\lambda_2(\mathcal{L}) = \min_{x\neq 0, \mathbf{1}_N^T x=0} \frac{x^T \mathcal{L} x}{x^T x}$.

Lemma 2 ([16]): Consider a continuous function $W(t) : [0,\infty) \to [0,\infty)$ and a timing sequence $\{t_s \in [0,\infty) | s \in \mathbb{N}\}$ satisfying $T_s \triangleq t_{s+1} - t_s \geq h$ for some $h \in \mathbb{R}_+$. If $W(t)$ is differentiable over every interval $[t_s, t_{s+1})$ and

$$\dot{W}(t) \leq -\beta_1^s W(t) + \beta_2^s W(t_s), \qquad \forall t \in [t_s, t_{s+1}),$$

with scalars $\beta_1^s > \beta_2^s > 0$, then $W(t)$ decays to zero, i.e., $\lim_{t\to\infty} W(t) = 0$. Here, the superscript s indicates that β_1^s, β_2^s need not to be uniform among different sampling intervals.

III. Distributed Aperiodic Time-Triggered Consensus: Linear and Lipschitz Nonlinear Cases

In this section, we study the following DATT consensus controller

$$u_i(t) = cK\delta_i(t_s), \quad t \in [t_s, t_{s+1}), \tag{3}$$

where $\delta_i(t_s)$ is as in (1), $c \in \mathbb{R}_+$ is the coupling gain of the agents and K is a constant gain matrix. We start with the linear case.

A. DATT: Linear Case

Consider the linear agents

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathcal{V} \triangleq \{1, 2, \ldots, N\}, \tag{4}$$

where $x_i \in \mathbb{R}^n$ is the state of agent i, and $u_i \in \mathbb{R}^m$ is its control input to be designed. Let the pair (A, B) be stabilizable, which is necessary and sufficient for the existence of a solution $P > 0$ to the following LMI, for some $\mu_1, \mu_2 \in \mathbb{R}_+$:

$$AP + PA^T - \mu_1 BB^T + \mu_2 P < 0. \tag{5}$$

The following scalable result holds for controller (3).

Theorem 1: Under Assumption 1, denote

$$\underline{c} = \frac{\mu_1}{2\lambda_2(\mathcal{L})}, \qquad \bar{T} = \sqrt{\frac{\alpha_1}{\alpha_2}}$$

with

$$\alpha_1 = \frac{\mu_2 \lambda_1(P)}{2\lambda_n(P)}, \qquad \alpha_2 = \frac{\xi_1^2 \xi_2^2 \lambda_n^2(P)}{2\mu_2},$$
$$\xi_1 = 2c\lambda_N(\mathcal{L})\lambda_n(P^{-1}BB^TP^{-1}),$$
$$\xi_2 = \lambda_n(A) + c\lambda_N(\mathcal{L})\lambda_n(BB^TP^{-1}).$$

Consider the feedback gain $K = -B^TP^{-1}$, the coupling gain $c \geq \underline{c}$, and the sampling interval $T_s < \bar{T}$, $\forall s$. Then, the MAS (4) reaches consensus under the DATT controller (3).

Proof: Let $e_i = x_i - \bar{x}$ where $\bar{x} = \frac{1}{N}\sum_{j=1}^N x_j$, $x = \mathrm{col}(x_1, x_2, \ldots, x_N)$, $e = \mathrm{col}(e_1, e_2, \ldots, e_N)$. Then, $e = (\varXi \otimes \mathbf{I}_n)x$ with $\varXi = \mathbf{I}_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. Under Assumption 1, it is easy to verify that $\mathcal{L}\varXi = \varXi\mathcal{L} = \mathcal{L}$, and $e = 0$ amounts to the consensus of x_i.

By (4) and (3), the closed-loop dynamics of x is

$$\dot{x} = (\mathbf{I}_N \otimes A)x + (c\mathcal{L} \otimes BK)x(t_s), \quad t \in [t_s, t_{s+1}).$$

Denote $\tilde{e}(t) = e(t_s) - e(t)$ for $t \in [t_s, t_{s+1})$. Then, the dynamics of e is

$$\begin{aligned}\dot{e} &= (\mathbf{I}_N \otimes A)e + (c\mathcal{L} \otimes BK)e(t_s)\\ &= (\mathbf{I}_N \otimes A + c\mathcal{L} \otimes BK)e + (c\mathcal{L} \otimes BK)\tilde{e},\\ &\quad t \in [t_s, t_{s+1}).\end{aligned} \tag{6}$$

It is clear that $e = 0$ iff all x_i reach consensus. During any sampling interval $t \in [t_s, t_{s+1})$, consider the candidate Lyapunov function

$$V_1(e(t)) = e^T(t)(\mathbf{I}_N \otimes P^{-1})e(t). \tag{7}$$

Then, it is clear that

$$\frac{\|e\|^2}{\lambda_n(P)} \leq V_1 \leq \frac{\|e\|^2}{\lambda_1(P)}. \tag{8}$$

Along the trajectory of (6) and using $K = -B^TP^{-1}$, the time derivative of V_1 is

$$\begin{aligned}\dot{V}_1 &= 2e^T(\mathbf{I}_N \otimes P^{-1}A - c\mathcal{L} \otimes P^{-1}BB^TP^{-1})e\\ &\quad - 2ce^T(\mathcal{L} \otimes P^{-1}BB^TP^{-1})\tilde{e}.\end{aligned}$$

By Lemma 1, there exists a unitary matrix $U = [\frac{\mathbf{1}}{\sqrt{N}}, \tilde{U}]$ such that $U^T\mathcal{L}U = \Lambda \triangleq \mathrm{diag}(0, \lambda_2(\mathcal{L}), \ldots, \lambda_N(\mathcal{L}))$. Let $\hat{e} = \mathrm{col}(\hat{e}_1, \ldots, \hat{e}_N) = (U^T \otimes P^{-1})e$, then $\hat{e}_1 = (\frac{\mathbf{1}_N^T}{\sqrt{N}} \otimes P^{-1})e = 0$. Note that $c \geq \frac{\mu_1}{2\lambda_2(\mathcal{L})}$, we have

$$\begin{aligned}\dot{V}_1 &= \hat{e}^T(\mathbf{I}_N \otimes (AP + PA^T) - 2c\Lambda \otimes BB^T)\hat{e}\\ &\quad - 2ce^T(\mathcal{L} \otimes P^{-1}BB^TP^{-1})\tilde{e}\\ &= \sum_{i=2}^N \hat{e}_i^T(AP + PA^T - 2c\lambda_i(\mathcal{L})BB^T)\hat{e}_i\\ &\quad - 2ce^T(\mathcal{L} \otimes P^{-1}BB^TP^{-1})\tilde{e}\\ &\leq -\mu_2 V_1 + \xi_1\|e\|\|\tilde{e}\| \leq -\frac{\mu_2}{2\lambda_n(P)}\|e\|^2 + \frac{\lambda_n(P)\xi_1^2}{2\mu_2}\|\tilde{e}\|^2\end{aligned} \tag{9}$$

where the first inequality uses (5), compatibility of matrix norm, and the definition of $\|\hat{e}\|$, while the last inequality uses (8) and Young's inequality.

For any $t \in [t_s, t_{s+1})$, applying the mean value theorem to e over period $[t_s, t]$, there must exist a $\tau_s \in (t_s, t) \subset [t_s, t_{s+1})$, such that $e(t) - e(t_s) = \dot{e}(\tau_s)(t - t_s)$. Then,

$$\|\tilde{e}(t)\| = \|e(t_s) - e(t)\| \leq \|\dot{e}(\tau_s)\|(t - t_s) \leq T_s\|\dot{e}(\tau_s)\|. \tag{10}$$

Further, note by (6) and (8) that, $\forall t \in [t_s, t_{s+1})$,

$$\begin{aligned}\|\dot{e}(t)\| &\leq \lambda_n(A)\|e\| + c\lambda_N(\mathcal{L})\lambda_n(BB^TP^{-1})\|e(t_s)\| \\ &\leq \lambda_n(A)\sqrt{\lambda_n(P)}\sqrt{V_1(t)} \\ &\quad + c\lambda_N(\mathcal{L})\lambda_n(BB^TP^{-1})\sqrt{\lambda_n(P)}\sqrt{V_1(t_s)} \\ &\leq \xi_2\sqrt{\lambda_n(P)}\sqrt{V_1^s(t)},\end{aligned} \tag{11}$$

where $V_1^s(t) = \max_{\tau\in[t_s,t]} V_1(\tau)$. It follows from (8)-(11) that

$$\begin{aligned}\dot{V}_1(t) &\leq -\frac{\mu_2}{2\lambda_n(P)}\|e(t)\|^2 + \frac{\lambda_n(P)\xi_1^2}{2\mu_2}T_s^2\|\dot{e}(\tau_s)\|^2 \\ &\leq -\alpha_1 V_1(t) + \alpha_2 T_s^2 V_1^s(t).\end{aligned} \tag{12}$$

Next, we claim that

$$V_1^s(t) = V_1(t_s) \qquad \forall t \in [t_s, t_{s+1}). \tag{13}$$

We prove the statement by contradiction. If (13) is not true, there must exist a $t' \in [t_s, t_{s+1})$ such that $V_1(t') > V_1(t_s)$. Note by (9) that $\dot{V}_1(t_s) \leq -\frac{\mu_2}{2\lambda_n(P)}\|e(t_s)\|^2$. Two cases will be considered:

Case 1: $e(t_s) = 0$. It follows by (6) that $e(t) \equiv 0$, $\forall t \geq t_s$, and there exists no such $t' \in [t_s, t_{s+1})$ with $V_1(t') > V_1(t_s)$, leading to a contradiction. In fact, in this case, the consensus problem is solved.

Case 2: $e(t_s) \neq 0$. There holds $\dot{V}_1(t_s) \leq 0$, implying that V_1 will decrease over a possibly short time interval from t_s. On the one hand, the continuity of V_1 guarantees that there must exist another $t'' \in (t_s, t')$ such that: $V_1(t_s) = V_1(t'')$; $V_1(t) \leq V_1(t'')$, $\forall t \in [t_s, t'']$; and $\dot{V}_1(t'') > 0$. On the other hand, provided $T_s < \sqrt{\frac{\alpha_1}{\alpha_2}}$, (12) guarantees that $\dot{V}_1(t'') \leq -\alpha_1 V_1(t'') + \alpha_2 T_s^2 V_1(t'') < 0$, which is a contradiction. Thus, the statement (13) holds true.

Now, by (12) and (13), we conclude that

$$\dot{V}_1(t) \leq -\alpha_1 V_1(t) + \alpha_2 T_s^2 V_1(t_s) \quad \forall t \in [t_s, t_{s+1}).$$

It follows from Lemma 2 and $0 < T_s < \sqrt{\frac{\alpha_1}{\alpha_2}}$ that $\lim_{t\to\infty} V_1(t) = 0$, i.e., $\lim_{t\to\infty} e(t) = 0$. This completes the proof. ■

Remark 2: Theorem 1 requires only two agent-level stabilization LMIs ((5) and $P > 0$, both with dimension n): thus, the framework is scalable to large networks. Note that [16] has recently proposed scalable LMI conditions in leader-following networks of linear agents. Differently, Theorem 1 solves the leaderless case by carefully addressing the semi-definiteness of the Laplacian matrix. Besides, in the following we will show that the proposed method can be naturally generalized along two directions, i.e., the case with Lipschitz nonlinear agents (Section III-B) and the case with event-triggered communication mechanism (Section IV).

B. DATT: Lipschitz Nonlinear Case

In this subsection, we consider the case of Lipschitz nonlinear MASs, that is

$$\dot{x}_i(t) = Ax_i(t) + f(x_i) + Bu_i(t), \quad i \in \mathcal{V}, \tag{14}$$

where the nonlinear function $f(\cdot)$ is ϕ-Lipschitz with $\phi \in \mathbb{R}_+$, i.e.,

$$\|f(x) - f(y)\| \leq \phi\|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{15}$$

Let us first solve the following LMI

$$\begin{pmatrix} AQ + QA^T - \nu_1 BB^T + \nu_2 Q + \nu_3 \mathbf{I}_n & Q \\ Q & -\phi^{-2}\nu_3\mathbf{I}_n \end{pmatrix} < 0 \tag{16}$$

to get a $Q > 0$ and $\nu_1, \nu_2, \nu_3 \in \mathbb{R}_+$. The following scalable result holds for the DATT controller (3).

Theorem 2: Under Assumption 1, denote

$$\underline{c}' = \frac{\nu_1}{2\lambda_2(\mathcal{L})}, \qquad \bar{T}' = \sqrt{\frac{\alpha_1'}{\alpha_2'}}$$

with

$$\begin{aligned}\alpha_1' &= \frac{\nu_2\lambda_1(Q)}{2\lambda_n(Q)}, \qquad \alpha_2' = \frac{\xi_1^2\xi_2^2\lambda_n^2(Q)}{2\nu_2}, \\ \xi_1' &= 2c\lambda_N(\mathcal{L})\lambda_n(Q^{-1}BB^TQ^{-1}), \\ \xi_2' &= \lambda_n(A) + 2\sqrt{N}\phi + c\lambda_N(\mathcal{L})\lambda_n(BB^TQ^{-1}).\end{aligned}$$

Consider the feedback gain $K = -B^TQ^{-1}$, the coupling gain $c \geq \underline{c}'$, and the sampling interval $T_s < \bar{T}'$, $\forall s$. Then, the MAS (14) reaches consensus under the DATT controller (3).

Proof: The same notations as in the proof of Theorem 1 will be inherited when no conflict arises. Based on (14) and (3), the closed-loop dynamics of e can be obtained as

$$\begin{aligned}\dot{e} &= (\mathbf{I}_N \otimes A)e + (\varXi \otimes \mathbf{I}_n)F(x) + (c\mathcal{L} \otimes BK)e(t_s) \\ &= (\mathbf{I}_N \otimes A + c\mathcal{L} \otimes BK)e + (\varXi \otimes \mathbf{I}_n)F(x) \\ &\quad + (c\mathcal{L} \otimes BK)\tilde{e}, \quad t \in [t_s, t_{s+1})\end{aligned} \tag{17}$$

where $F(x) = \mathrm{col}(f(x_1), f(x_2), \ldots, f(x_N))$.

During any sampling interval $t \in [t_s, t_{s+1})$, consider the candidate Lyapunov function

$$V_2(e(t)) = e^T(t)(\mathbf{I}_N \otimes Q^{-1})e(t).$$

Then, we have

$$\frac{\|e\|^2}{\lambda_n(Q)} \leq V_2 \leq \frac{\|e\|^2}{\lambda_1(Q)}. \tag{18}$$

Along the trajectory of (6) and using $K = -B^TQ^{-1}$, the time derivative of V_2 is

$$\begin{aligned}\dot{V}_2 &= 2e^T(\mathbf{I}_N \otimes Q^{-1}A - c\mathcal{L} \otimes Q^{-1}BB^TQ^{-1})e \\ &\quad + 2e^T(\varXi \otimes Q^{-1})F(x) - 2ce^T(\mathcal{L} \otimes Q^{-1}BB^TQ^{-1})\tilde{e}.\end{aligned} \tag{19}$$

Note that

$$\begin{aligned}&2e^T(\Xi\otimes Q^{-1})F(x)\\&=2\sum_{i=1}^N e_i^TQ^{-1}(f(x_i)-f(\bar{x})+f(\bar{x})-\frac{1}{N}\sum_{j=1}^N f(x_j))\\&\le 2\sum_{i=1}^N \phi\|Q^{-1}e_i\|\|e_i\|\\&\le e^T(\mathbf{I}_N\otimes(\nu_3^{-1}\phi^2+\nu_3(Q^{-1})^2))e\end{aligned}\tag{20}$$

where the first inequality uses (15) and the fact that $(\mathbf{1}_N\otimes\mathbf{I}_n)e=0$, while the last inequality uses Young's inequality.

Similarly to (9), let us denote $\check{e}=\mathrm{col}(\check{e}_1,\ldots,\check{e}_N)=(U^T\otimes Q^{-1})e$, then $\check{e}_1=(\frac{\mathbf{1}_N^T}{\sqrt{N}}\otimes Q^{-1})e=0$. Noting that $c\ge\frac{\nu_1}{2\lambda_2(\mathcal{L})}$, it follows from (19)-(20) that

$$\begin{aligned}\dot{V}_2&=\check{e}^T(\mathbf{I}_N\otimes(AQ+QA^T)-2c\Lambda\otimes BB^T)\check{e}\\&\quad+\check{e}^T(\mathbf{I}_N\otimes(\nu_2^{-1}\phi^2Q^2+\nu_2\mathbf{I}_n))\check{e}\\&\quad-2ce^T(\mathcal{L}\otimes Q^{-1}BB^TQ^{-1})\tilde{e}\\&=\sum_{i=2}^N\check{e}_i\big(AQ+QA^T-2c\lambda_i(\mathcal{L})BB^T+\nu_3^{-1}\phi^2Q^2\\&\quad+\nu_3\mathbf{I}_n)\check{e}_i-2ce^T(\mathcal{L}\otimes Q^{-1}BB^TQ^{-1})\tilde{e}\\&\le-\nu_2V_2+\xi_1'\|e\|\|\tilde{e}\|\\&\le-\frac{\nu_2}{2\lambda_n(Q)}\|e\|^2+\frac{\lambda_n(Q)\xi_1'^2}{2\nu_2}\|\tilde{e}\|^2\end{aligned}\tag{21}$$

where the LMI (16) and Schur complement [32] have been used to obtain the first inequality.

Note that (10) still holds. Differently, it follows from (17) and (18) that, $\forall t\in[t_s,t_{s+1})$,

$$\begin{aligned}\|\dot{e}(t)\|\le\lambda_n(A)\|e\|+\|(\Xi\otimes\mathbf{I}_n)F(x)\|\\+c\lambda_N(\mathcal{L})\lambda_n(BB^TQ^{-1})\|e(t_s)\|.\end{aligned}\tag{22}$$

Clearly, we need to establish an upper bound of the nonlinear term $\|(\Xi\otimes\mathbf{I}_n)F(x)\|$ in (22). For notational convenience, let us denote $\Psi(x)=(\Xi\otimes\mathbf{I}_n)F(x)$, and define $\eta=(U^T\otimes\mathbf{I}_n)e$ according to the unitary matrix U as before. If we partition the state of $\Psi(x)$ as $\Psi(x)=\mathrm{col}(\psi_1(x),\ldots,\psi_N(x))$ where $\psi_i(\cdot):\mathbb{R}^{Nn}\to\mathbb{R}^n$, then

$$\psi_i(x)=f(x_i)-\frac{1}{N}\sum_{j=1}^N f(x_j)=\frac{1}{N}\sum_{j=1}^N(f(x_i)-f(x_j)).$$

By the triangular inequality of norms and the Lipschitz condition (15), there holds

$$\|\psi_i(x)\|\le\frac{1}{N}\sum_{j=1}^N\|f(x_i)-f(x_j)\|\le\frac{\phi}{N}\sum_{j=1}^N\|x_i-x_j\|\tag{23}$$

where ϕ is the Lipschitz coefficient in (15). Note that

$$x_i-x_j=e_i-e_j=((r_i-r_j)\otimes\mathbf{I}_n)\eta$$

where r_i denotes the i-th row of U. Moreover, since U is unitary, $\|r_i\|=1,\forall i$. Then, it follows from (23) that

$$\|\psi_i(x)\|\le\frac{\phi}{N}\sum_{j=1}^N(\|r_i\|+\|r_j\|)\|\eta\|=2\phi\|\eta\|.$$

Thus, we have

$$\begin{aligned}\|(\Xi\otimes\mathbf{I}_n)F(x)\|=\|\Psi(x)\|=\sqrt{\sum_{i=1}^N\|\psi_i(x)\|^2}\\\le\sqrt{\sum_{i=1}^N4\phi^2\|\eta\|^2}=2\sqrt{N}\phi\|\eta\|=2\sqrt{N}\phi\|e\|.\end{aligned}\tag{24}$$

Now, based on (22) and (24),

$$\begin{aligned}\|\dot{e}(t)\|&\le\lambda_n(A)\|e\|+2\sqrt{N}\phi\|e\|\\&\quad+c\lambda_N(\mathcal{L})\lambda_n(BB^TQ^{-1})\|e(t_s)\|\\&\le(\lambda_n(A)+2\sqrt{N}\phi)\sqrt{\lambda_n(Q)}\sqrt{V_2(t)}\\&\quad+c\lambda_N(\mathcal{L})\lambda_n(BB^TQ^{-1})\sqrt{\lambda_n(Q)}\sqrt{V_2(t_s)}\\&\le\xi_2'\sqrt{\lambda_n(Q)}\sqrt{V_2^s(t)},\end{aligned}\tag{25}$$

where $V_2^s(t)=\max_{\tau\in[t_s,t]}V_2(\tau)$. Then, it follows from (18), (21), (10) and (25) that

$$\begin{aligned}\dot{V}_2(t)&\le-\frac{\nu_2}{2\lambda_n(Q)}\|e(t)\|^2+\frac{\lambda_n(Q)\xi_1'^2}{2\nu_2}T_s^2\|\dot{e}(\tau_s)\|^2\\&\le-\alpha_1'V_2(t)+\alpha_2'T_s^2V_2^s(t).\end{aligned}$$

Following similar steps as in the proof of Theorem 1, we conclude that

$$\dot{V}_2(t)\le-\alpha_1'V_2(t)+\alpha_2'T_s^2V_2(t_s)\quad\forall t\in[t_s,t_{s+1}).$$

Then, it follows from Lemma 2 and $0<T_s<\sqrt{\frac{\alpha_1'}{\alpha_2'}}$ that $\lim_{t\to\infty}V_2(t)=0$, i.e., $\lim_{t\to\infty}e(t)=0$. ■

Several remarks are readily presented.

Remark 3: Based on Finsler's lemma [32], the feasibility of (16) is equivalent to the existence of a matrix E and ν_3 such that $(A-BE)Q+Q(A-BE)^T+\nu_3\mathbf{I}_n+\nu_3^{-1}\phi^2Q^2<0$, provided B has full rank. In the case of $\nu_2=1$, the feasibility of (16) is dual to the existence of an observer for a single ϕ-Lipschitz nonlinear system [33], [34].

Remark 4: Distributed periodic time-triggered consensus of Lipschitz nonlinear MASs has been addressed via the Lyapunov functional method in the literature: there, significantly more and larger LMIs conditions (linear/polynomial on N) are required, see e.g., [4], [8], [9]. Here, a different and simpler Lyapunov method has been adopted by directly analyzing the piecewise continuous closed-loop dynamics. Only two

LMIs ((16) of dimension $2n$ and $Q > 0$ of dimension n) are required, which enables great scalability in large networks.

Remark 5: It is not hard to see that the derived maximum sampling interval (in Theorem 1 and 2) is positively related to $\frac{\lambda_2(\mathcal{L})}{\lambda_N(\mathcal{L})}$. This implies that, roughly speaking, denser communication graph allows larger sampling interval, which is consistent with intuition, and also confirms the results in [15]. It should be mentioned that the estimation requires some global knowledge, i.e., the eigenvalues of the Laplacian matrix, and the agent cardinality in the nonlinear case. In fact, such knowledge is widely needed in sampled-data based consensus solutions, whether the stability analysis is based on Lyapunov functional method [3], [4], [5], [6], [7], [8], [9], or the decomposition idea [15], [16]. In practice, if not a priori available, this knowledge can be reconstructed in a distributed way by the agents themselves, see e.g. [35], [36].

IV. Distributed Aperiodic Event-Triggered Consensus: Linear and Lipschitz Nonlinear Cases

In this section, we show how the proposed DATT consensus in Section III can be naturally generalized into an event-based setting, i.e., being DAET. Consider the following controller

$$u_i(t) = cK\hat{\delta}_i(t_s), \quad t \in [t_s, t_{s+1}), \tag{26}$$

where $\hat{\delta}_i(t_s)$ is defined as in (2). In DAET, each agent $i \in \mathcal{V}$ maintains an event detector defined as

$$E_i(t_s, \epsilon_i(t_s)) : \|\epsilon_i(t_s)\|^2 \geq H_i(t_s) \text{ or } \delta_i(t_s) = 0 \tag{27}$$

where $\epsilon_i(t_s) = \hat{x}_i(t_s) - x_i(t_s)$, and $H_i(t)$ is a continuous threshold function that satisfies:

$$H_i(t) \in \mathbf{L}^\infty,\ H_i(t) \geq 0 \text{ for all } t \geq 0,\ \lim_{t\to\infty} H_i(t) = 0. \tag{28}$$

As we discussed before, agent i broadcasts its current sampled state $x_i(t_s)$ to its neighbors (and its own controller) only if E_i is triggered.

Before giving the main results of this section, we first generalize Lemma 2 as follows.

Lemma 3: Consider a continuous function $W(t) : [0, \infty) \to [0, \infty)$ and a timing sequence $\{t_s \in [0, \infty) | s \in \mathbb{N}\}$ satisfying $T_s \triangleq t_{s+1} - t_s \geq h$ for some $h \in \mathbb{R}_+$. Suppose $W(t)$ is differentiable over every interval $[t_s, t_{s+1})$ and

$$\dot{W}(t) \leq -\beta_1^s W(t) + \beta_2^s W(t_s) + Y(t_s), \quad \forall t \in [t_s, t_{s+1}),$$

with scalars $\beta_1^s > \beta_2^s > 0$. Here, $Y(t)$ is a continuous function that satisfies:

$$Y(t) \in \mathbf{L}^\infty,\ Y(t) \geq 0 \text{ for all } t \geq 0,\ \lim_{t\to\infty} Y(t) = 0.$$

Then $W(t)$ decays to zero, i.e., $\lim_{t\to\infty} W(t) = 0$. Here, the superscript s indicates that β_1^s, β_2^s need not to be uniform among different sampling intervals.

Proof: See Appendix. ■

A. DAET: Linear Case

Consider the linear MAS (4) under the scalable LMI condition (5). We have the following result for the DAET controller (26).

Theorem 3: Under Assumption 1, denote

$$\underline{c} = \frac{\mu_1}{2\lambda_2(\mathcal{L})}, \qquad \bar{T}_\mathrm{e} = \frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}}$$

with α_1, α_2 the same as in Theorem 1. Consider the feedback gain $K = -B^T P^{-1}$, the coupling gain $c \geq \underline{c}$, and the sampling interval $T_s < \bar{T}_\mathrm{e}$, $\forall s$. Then, the MAS (4) reaches consensus under the DAET controller (26).

Proof: The same notations as in the proof of Theorem 1 will be inherited when no conflict arise. Denote $\hat{e}(t_s) = (\Xi \otimes \mathbf{I}_n)\hat{x}(t_s)$. Then, the dynamics of the consensus error system e follows

$$\begin{aligned}\dot{e} &= (\mathbf{I}_N \otimes A)e + (c\mathcal{L} \otimes BK)\hat{e}(t_s) \\ &= (\mathbf{I}_N \otimes A + c\mathcal{L} \otimes BK)e \\ &\quad + (c\mathcal{L} \otimes BK)(\tilde{e} + \epsilon(t_s)), \quad t \in [t_s, t_{s+1}).\end{aligned} \tag{29}$$

where $\epsilon(t_s) = \mathrm{col}(\epsilon_1(t_s), \ldots, \epsilon_N(t_s))$. During any interval $t \in [t_s, t_{s+1})$, consider the candidate Lyapunov function V_1 (7). Following similar procedure to get (9), we have

$$\dot{V}_1 \leq -\frac{\mu_2}{2\lambda_n(P)}\|e\|^2 + \frac{\lambda_n(P)\xi_1^2}{2\mu_2}\|\tilde{e} + \epsilon(t_s)\|^2 \tag{30}$$

provided that $c \geq \underline{c}$. By the first half of the event condition (27), there holds $\|\epsilon_i(t_s)\|^2 \leq H_i(t_s)$. Thus,

$$\dot{V}_1 \leq -\frac{\mu_2}{2\lambda_n(P)}\|e\|^2 + \frac{\lambda_n(P)\xi_1^2}{\mu_2}\|\tilde{e}\|^2 + \frac{\lambda_n(P)\xi_1^2}{\mu_2}\sum_{i=1}^N H_i(t_s). \tag{31}$$

Since $\|\tilde{e} + \epsilon(t_s)\| \leq \|\tilde{e}\| + \|\epsilon(t_s)\|$, we can follow a similar procedure that used to get (11), leading to

$$\begin{aligned}\|\dot{e}(t)\| &\leq \xi_2\sqrt{\lambda_n(P)}\sqrt{V_1^s(t)} \\ &\quad + c\lambda_N(\mathcal{L})\lambda_n(BB^T P^{-1})\sqrt{\sum_{i=1}^N H_i(t_s)},\end{aligned} \tag{32}$$

Note that (10) still holds, which allows to write

$$\begin{aligned}\dot{V}_1 \leq &-\frac{\mu_2}{2\lambda_n(P)}\|e\|^2 + \frac{\lambda_n(P)\xi_1^2}{\mu_2}T_s^2 \\ &\left(2\xi_2^2\lambda_n(P)V_1^s(t) + 2c^2\lambda_N^2(\mathcal{L})\lambda_n^2(BB^T P^{-1})\sum_{i=1}^N H_i(t_s)\right) \\ &+ \frac{\lambda_n(P)\xi_1^2}{\mu_2}\sum_{i=1}^N Hi(t_s) \\ \leq &-\alpha_1 V_1(t) + 4\alpha_2 T_s^2 V_1^s(t) + \zeta\sum_{i=1}^N H_i(t_s)\end{aligned} \tag{33}$$

where $\zeta = \frac{4\lambda_N^2(\mathcal{L})\lambda_n^2(BB^TP^{-1})T_s^2}{\xi_2^2}\alpha_2$.

Next, we claim that there exists a $s^* \in \mathbb{N}$, such that

$$V_1^s(t) = V_1(t_s) \qquad \forall t \in [t_s, t_{s+1}),\ \forall s > s^* \tag{34}$$

We prove the statement by contradiction. If (34) is not true, then $\forall s_0 \in \mathbb{N}$, one can always find a $s_1 > s_0$, such that there exists a $t' \in [t_{s_1}, t_{s_1+1})$ with $V_1(t') > V_1(t_{s_1})$. Note by (31) that

$$\dot{V}_1(t_{s_1}) \leq -\frac{\mu_2}{2\lambda_n(P)}\|e(t_{s_1})\|^2 + \frac{\lambda_n(P)\xi_1^2}{\mu_2}\sum_{i=1}^{N} H_i(t_{s_1}). \tag{35}$$

We consider the following two cases.

Case 1: $e(t_{s_1}) = 0$. There holds $\delta_i(t_{s_1}) = (\mathcal{L} \otimes \mathbf{I}_n)e(t_{s_1}) = 0$. It follows by the second half of the event condition (27) and (29) that $e(t) \equiv 0$, $\forall t \geq t_{s_1}$, and there exists no such t' with $V_1(t') > V_1(t_{s_1})$, leading to a contradiction. In fact, in this case, the consensus problem is solved, the event will be triggered for any agent at any sampled instant, i.e., the DAET control degenerates to the DATT control.

Case 2: $e(t_{s_1}) \neq 0$. It is clear that the function $\sum_{i=1}^{N} H_i(\cdot)$ also satisfies the condition (28) (as the number of the agents are finite). Then for $\epsilon_1 = \frac{\mu_2^2}{4\lambda_n^2(P)\xi_1^2}\|e(t_{s_1})\|^2 > 0$, there exists a $s_2 \in \mathbb{N}$, such that $\forall s > s_2$, there holds $\sum_{i=1}^{N} H_i(t_s) < \epsilon_1$. Let $s_0 > s_2$, then $s_1 > s_2$. It follows from (35) that $\dot{V}_1(t_{s_1}) \leq -\frac{\mu_2}{4\lambda_n(P)}\|e(t_{s_1})\|^2 < 0$, implying that V_1 will decrease over a possibly short time interval from t_{s_1}.

On the one hand, the continuity of V_1 guarantees that there must exist another $t'' \in (t_{s_1}, t')$ such that: $V_1(t_{s_1}) = V_1(t'')$; $V_1(t) \leq V_1(t'')$, $\forall t \in [t_{s_1}, t'']$; and $\dot{V}_1(t'') \geq q > 0$ for some q. On the other hand, provided $T_s < \frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}}$, (33) guarantees that $\dot{V}_1(t'') \leq -\alpha_1 V_1(t'') + 4\alpha_2\, T_s^2 V_1(t'') + \zeta\sum_{i=1}^{N} H_i(t_{s_1}) < \zeta\sum_{i=1}^{N} H_i(t_{s_1})$. Similarly, there exists a $s_3 \in \mathbb{N}$, such that $\forall s > s_3$, there holds $\sum_{i=1}^{N} H_i(t_s) < \frac{q}{\zeta}$. Let $s_0 > s_3$, then $s_1 > s_3$, which implies that $\dot{V}_1(t'') < q$, leading to a contradiction.

Combing the above two cases, the statement (34) holds true. Now, we know that there exists a $s^* \in \mathbb{N}$, such that $\forall s > s^*$,

$$\dot{V}_1 \leq -\alpha_1 V_1 + 4\alpha_2 T_s^2 V_1(t_s) + \zeta\sum_{i=1}^{N} H_i(t_s) \quad \forall t \in [t_s, t_{s+1}).$$

Applying Lemma 3 from the sampling instant s^* guarantees the convergence of $V_1(t)$ to zero, i.e., the agents reach consensus. ■

B. DAET: Lipschitz Nonlinear Case

Consider the Lipschitz nonlinear MAS (14) under the scalable LMI condition (16). We have the following result for the DAET controller (26).

Theorem 4: Under Assumption 1, denote

$$\underline{c}' = \frac{\nu_1}{2\lambda_2(\mathcal{L})}, \qquad \bar{T}_{\mathrm{e}}' = \frac{1}{2}\sqrt{\frac{\alpha_1'}{\alpha_2'}}$$

with α_1', α_2' the same as in Theorem 2. Consider the feedback gain $K = -B^TQ^{-1}$, the coupling gain $c \geq \underline{c}'$, and the sampling interval $T_s < \bar{T}_{\mathrm{e}}'$, $\forall s$. Then, the MAS (14) reaches consensus under the DAET controller (26).

Proof: The proof can be performed by combing the proof of Theorem 2 with that of Theorem 3. The details are omitted to avoid repetition. ■

Remark 6: Note that the estimate of the maximum sampling interval in the DAET case is half the estimate in the DATT case. This is essentially due to the need to apply extra norm inequalities in (30) and (32). Apart from this, the main difference between the proofs of Theorem 1 and Theorem 3 is the following: the Lyapunov function is non-increasing during $[t_s, t_{s+1})$ and $\forall s$ in the DATT case, while it is non-increasing during $[t_s, t_{s+1})$ and $\forall s > s^*$ for some s^* in the DAET case, i.e., only when sufficient samplings are taken. This means that the DAET control allows the consensus error to grow temporarily (during the sampling intervals), while suppressing it more and more as the threshold functions decay to zero.

Remark 7: The design of the proposed DAET involves a class of $\mathbf{L}^\infty$ signals as threshold functions, which is inspired by [37]. Yet, our analysis is completely different from [37] since DAET is based on discrete-time communication. The main idea of the proof of Theorem 3 (and the proof of Lemma 3) is to generalize the scalable result of Theorem 1 with basic $\epsilon - \delta$ language in mathematical analysis. In practice, candidates of $H_i(t)$ in (28) include exponentially decaying functions or convergent and bounded functions such as $\frac{1}{t+1}$, etc. Generally speaking, the faster the threshold functions decay to zero, the better the convergence, at the price that more events will be triggered. Finally, we remark that the agents are allowed to choose their own threshold function $H_i(t)$.

Remark 8: The proof of Theorem 3 shows that as the agents reach consensus, the DAET control may degenerate to the DATT control to maintain consensus, which is reasonable. In fact, many event-based consensus controller in the literature cannot guarantee the Zeno-free property as the agents reach consensus (as $t \to \infty$) see e.g. [37], [38], [39]. From this perspective, the proposed DAET control naturally excludes Zeno behavior and also tries best to do this (with an estimation of the maximum sampling interval), which is extremely desired in practice.

V. Discussions

(*On extensions to switching graphs.*) All the main results in this paper including Theorems 1-4 can be extended to MASs with switching connected graphs after some minor modifications. For instance, we have the following corollary based upon Theorem 1:

Corollary 1: Suppose the communication graph is switching over a finite set of connected graphs $\{\mathcal{G}_j, j = 1, \ldots, l\}$. Denote

$$\underline{c} = \frac{\mu_1}{2\min\{\lambda_2(\mathcal{L}_j)\}}, \qquad \bar{T} = \sqrt{\frac{\alpha_1}{\alpha_2}}$$

with

$$\alpha_1 = \frac{\mu_2 \lambda_1(P)}{2\lambda_n(P)}, \qquad \alpha_2 = \frac{\xi_1^2 \xi_2^2 \lambda_n^2(P)}{2\mu_2},$$
$$\xi_1 = 2c\max\{\lambda_N(\mathcal{L}_j)\}\lambda_n(P^{-1}BB^T P^{-1}),$$
$$\xi_2 = \lambda_n(A) + c\max\{\lambda_N(\mathcal{L}_j)\}\lambda_n(BB^T P^{-1}).$$

Consider the feedback gain $K = -B^T P^{-1}$, the coupling gain $c \geq \underline{c}$, and the sampling interval $T_s < \bar{T}$, $\forall s$. Then, the MAS (4) reaches consensus under the DATT controller (3).

Similar corollaries can be written down also for Theorems 2-4. In fact, the extension relies on the fact that the Lyapunov candidate functions are independent of the graph Laplacian matrix $\mathcal{L}$. More details can be found in e.g. [16], [40].

(*On extensions to directed graphs.*) All the results in this work are for leaderless consensus with undirected graphs. Extension to directed graphs would be possible, provided we consider a leader-following setting along a similar analysis as [16]: however, we have preferred to consider the leaderless consensus setting, which is essentially more complex due to the lack of a reference leader in the network, as pointed out in e.g. [41].

(*On the parameters of the LMI conditions.*) The proposed DATT and DAET control, along with the estimations of the maximum sampling interval, rely on agent-lever stabilization via LMI conditions. Note that different LMI parameters (μ_1, μ_2 in (5) and ν_1, ν_2, ν_3 in (16)) would lead to different control gains, and thereby different estimations of the maximum sampling interval. In the linear case, we can write $\bar{T} = \frac{\mu_2}{\xi_1 \xi_2 \lambda_n(P)} \sqrt{\frac{\lambda_1(P)}{\lambda_n(P)}}$. Then, increasing μ_2 appropriately may lead to a larger $\bar{T}$, but increasing μ_2 too much would require a larger μ_1 by (5), and then larger ξ_1, ξ_2, finally leading to a smaller $\bar{T}$. Thanks to the explicit relations derived in Theorem 1-4, these parameters can be easily optimized via line search, grid search or other intelligent optimization algorithms. Similar arguments also apply to the nonlinear case where only one extra parameter ν_3 has been introduced to address the nonlinearity.

(*On 'continuously' monitoring in DAET.*) The events (27) in DAET control indicates that each agent i needs to 'continuously' (at every sampling instant, so not truly in a continuous-time fashion) monitor its neighboring states to evaluate $\delta_i(t_s)$. This problem widely exists in continuous event-triggered consensus [37], [39], and also in periodic event-triggered consensus [25]. A possible solution to relax this requirement is to generalize the idea of self-triggered control in continuous-time case [37], [39] to fit the proposed framework, which will be addressed in our future work.

VI. Numerical Examples

We will first present two examples for the linear case and Lipschitz nonlinear case, respectively. In each example, the communication graph is a Gilbert random graph $\mathcal{G}(N, p)$ with node number N and link probability $p \in (0, 1)$ [42]. Then, a third example is provided for comparison purpose. For numerical simulations, Runge-Kutta method with constant step size 0.01 (which can be seen as the lower bound h of the sampling intervals) is used, and the initial states of the agents are selected randomly according to a Gaussian distribution with standard deviation 5.

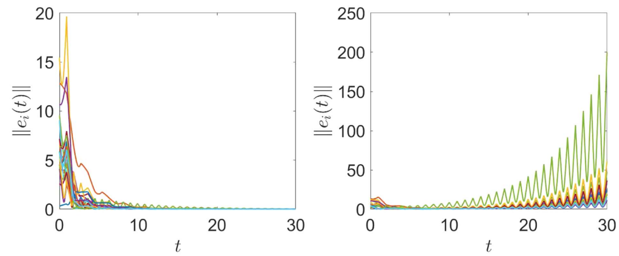

Fig. 1. Example 1: The consensus error $\|e_i(t)\|$ of 20 networked oscillators with DATT control under sampling period $T_s = 0.9$ (left), and $T_s = 1$ (right).

Example 1: Consider a group of 20 agents interacting over a Gilbert graph $\mathcal{G}(20, 0.5)$. The dynamics of the agents are second-order oscillators where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\mu_1 = 200$, $\mu_2 = 20$, solving the LMI (5) gives

$$P = \begin{pmatrix} 163.6887 & -40.0000 \\ -40.0000 & 163.6887 \end{pmatrix},$$

resulting in the feedback gain $K = (-0.0016, -0.0065)$. The graph $\mathcal{G}$ has $\lambda_2(\mathcal{L}) = 5.8327$ and $\lambda_N(\mathcal{L}) = 17.6704$. Following Theorem 1, one gets $\underline{c} = 17.1446$ and $\bar{T} = 0.9328$ for the DATT controller (3). Let $c = \underline{c}$ and $T_s = 0.9$. Then, the norms of the consensus errors $e_i(t) = x_i(t) - \frac{1}{N}\sum_{j=1}^N x_j(t)$ are shown in Fig. 1(left). Note that choosing $T_s = 1 > \bar{T}$ prevents the agents from reaching consensus, as shown in Fig. 1 (right). This indicates that the estimated maximum sampling interval $\bar{T}$ is quite tight.

By Theorem 3, we have $\bar{T}_e = 0.4664$ for the DAET controller (26) with the same parameters. Let $T_s = 0.4$ and $H_i(t) = 5\ \mathbf{exp}[-0.1t]$. The states of the agents are shown in Fig. 2, and the triggered instants are shown in Fig. 3(only the agents with even index are shown to make the graph more readable).

Example 2: The Chua's circuit network serves widely for testing chaos synchronization and cooperative control, see e.g. [5], [43], [44]. Let us consider a network of 100 Chua's circuit systems, interacting over a Gilbert graph $\mathcal{G}(100, 0.2)$. The dynamics of each circuit system is described as (14) with

$$A = \begin{pmatrix} 0.5598 & -1.3018 & 0 \\ 1 & -1 & 1 \\ 0 & 0.0135 & 0.0297 \end{pmatrix}, \quad B = \mathbf{I}_3,$$

and $f(x_i) = (0.442(|x_{i1} + 1| - |x_{i1} - 1|), 0, 0)^T$. The Lipschitz coefficient is $\phi = 2.9889$ [5].

Let $\nu_1 = 1000$, $\nu_2 = 396$, $\nu_3 = 175$, solving the LMI (16) gives

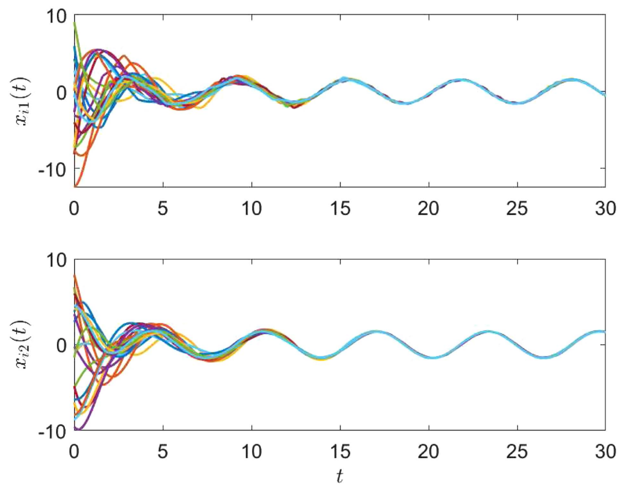

Fig. 2. Example 1: The states $x_i(t)$ of 20 networked oscillators with DAET control under sampling period $T_s = 0.4$.

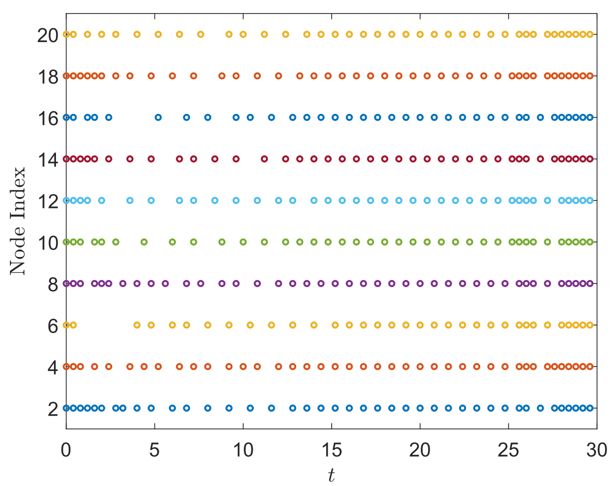

Fig. 3. Example 1: The triggered instants of the agents with even index.

$$Q = \begin{pmatrix} 68.5121 & -1.5524 & 0.5079 \\ -1.5524 & 65.8175 & 3.1387 \\ 0.5079 & 3.1387 & 70.4035 \end{pmatrix},$$

resulting in the feedback gain

$$K = \begin{pmatrix} -0.0146 & -0.0004 & 0.0001 \\ -0.0004 & -0.0152 & 0.0007 \\ 0.0001 & 0.0007 & -0.0142 \end{pmatrix}.$$

The graph $\mathcal{G}$ has $\lambda_2(\mathcal{L}) = 11.0831$ and $\lambda_N(\mathcal{L}) = 32.5336$. Following Theorem 2, one get $\underline{c}' = 45.1137$ and $\bar{T}' = 0.0841$ for the DATT controller (3). Let $c = \underline{c}'$ and $T_s = 0.08$. Then, the norms of $\|e_i(t)\|$ are shown in Fig. 4(left). Again, the estimated maximum sampling interval is quite tight since choosing $T_s = 0.09 > \bar{T}'$ prevents the circuit systems from reaching consensus, as shown in Fig. 4 (right).

From Theorem 4, we have $\bar{T}_e = 0.0421$ for the DAET controller (26) with the same parameters. Let $T_s = 0.04$ and $H_i(t) = \mathbf{exp}[-t]$. The evolution of the states of the agents is

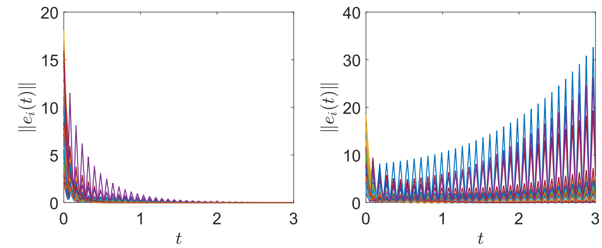

Fig. 4. Example 2: The consensus error $\|e_i(t)\|$ of 100 networked Chua's circuit systems with DATT control under sampling period $T_s = 0.08$ (left), and $T_s = 0.09$ (right).

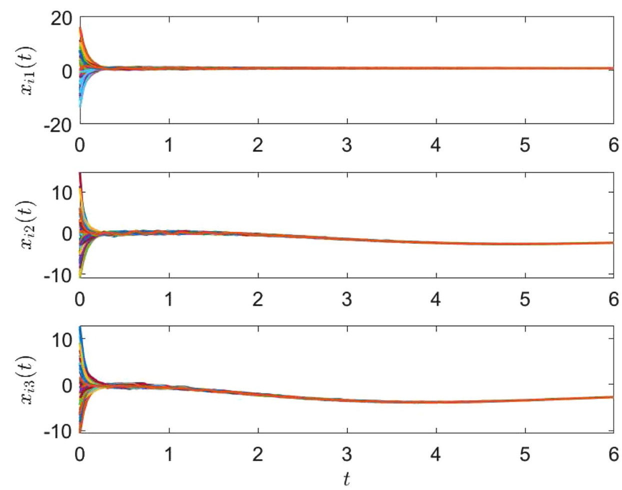

Fig. 5. Example 2: The states $x_i(t)$ of 100 networked Chua's circuit systems with DAET control under sampling period $T_s = 0.04$.

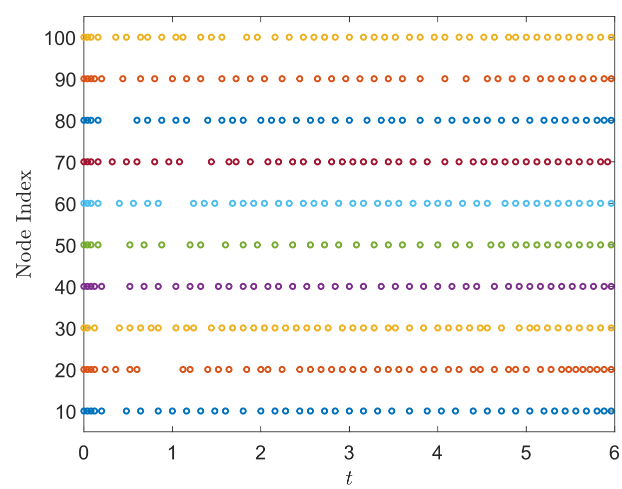

Fig. 6. Example 2: The triggered instants of the agents with index $10, 20, \ldots, 100$.

shown in Fig. 5, and the triggered instants of the agents with index $10, 20, \ldots, 100$ is shown in Fig. 6.

Example 3: For comparison purpose, let us consider 4 Chua's circuit systems above interacting over a graph $\mathcal{G}$ with edge set $\mathcal{E} = \{\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{34}, \mathcal{E}_{41}, \mathcal{E}_{24}\}$, same as [9]. In this case, $\lambda_2(\mathcal{L}) = 2$ and $\lambda_N(\mathcal{L}) = 4$. By Remark 5, it should be expected that a larger $\bar{T}'$ may be allowed as compared with Example 2. Let $\nu_1 = 600$, $\nu_2 = 220$, $\nu_3 = 95$, resulting in $\underline{c}' = 150$ and

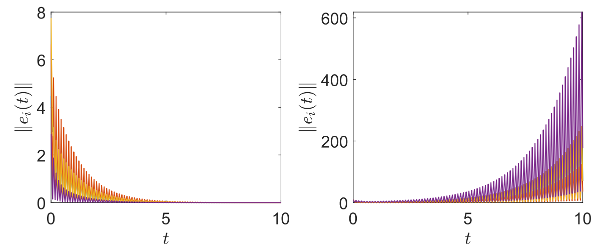

Fig. 7. Example 3: The consensus error $\|e_i(t)\|$ of 4 networked Chua's circuit systems with DATT control under sampling period $T_s = 0.11$ (left), and $T_s = 0.12$ (right).

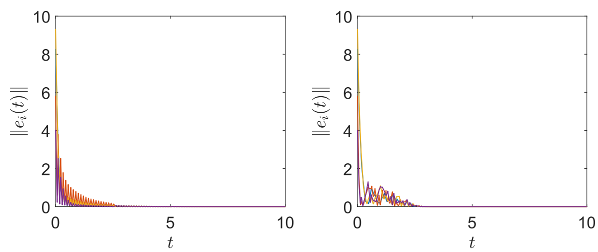

Fig. 8. Example 3 (switching graphs): The consensus error $e_i(t)$ with DATT control under sampling period $T_s = 0.11$ (left), and with DAET control under $T_s = 0.02$ when $t \leq 5$ and $T_s = 0.05$ thereafter (right).

$\bar{T}' = 0.1190$. The simulations under DATT control with $c = 150$, and $T_s = 0.11$ and 0.12 respectively, are shown in Fig. 7.

It should be noted that even though a larger $\bar{T}' = 0.33$ has been obtained in [9] for exponential consensus, the estimation is based on much weaker nonlinearities by assuming $\phi = 0.884$. Besides, as mentioned earlier, the input-delay method used therein may not be scalable to large networks, such as the above Example 2. More specifically, a total number of 2 N LMIs are needed with the dimension of decision variables being $12n^2 + mn$ in [9, Theorem 1], while only 2 LMIs are needed with the dimension of decision variables being $n^2 + 3$ in the proposed Theorem 2 ($Q > 0$ and $\nu_1, \nu_2, \nu_3 \in \mathbb{R}_+$).

To verify Corollary 1, assume the graph is randomly switching between the original graph $\mathcal{G}$ and the ring graph: this may model a randomly cyber-attack to the specific communication link $\mathcal{E}_{24}$. While this would not bring differences of our control strategy as both graphs share the same minimum and maximum eigenvalues. For comparisons between DATT and DAET, we first show the DATT consensus error with $T_s = 0.11$ in Fig. 8 (left), where the communication are triggered 91 times for each agent during the simulation interval (see Fig. 9). For DAET, to show the effectiveness of aperiodicity sampling, let $T_s = 0.02$ when $t \leq 5$, and $T_s = 0.05$ thereafter. Let $H_i(t) = \frac{3\sin^2(t)}{(t+0.01)^2}$. The evolution of the consensus errors $e_i(t)$ is shown in Fig. 8 (right). The triggered instants of the agents with DAET is shown in Fig. 10. Interestingly, even though DAET has a smaller sampling interval, the events are triggered with a total number of 69, 74, 75, 72 for the four agents during the simulation interval, without sacrificing the convergence rate: the settling time are nearly the same. This implies that DAET has the potential of further reducing costs of communication and control updates.

Finally, to show the impact of heterogeneous threshold functions, let us modify $H_i(t) = \frac{3\cos^2(t)}{(t+0.01)^4}$ for agent $i = \{1, 3\}$ in the above DAET case. Under the same other conditions, the events are triggered with a total number of 149, 73, 148, 76 for the four agents (Fig. 11), and the consensus errors are shown in Fig. 12. As compared with Fig. 8 (right), it can be seen that the heterogeneity of the threshold functions leads to larger transient consensus errors, even though there are more events triggered in the network. This is interesting and actually reasonable since the agents are not in a same rhythm, i.e., their willingness to communicate often disagrees with each other.

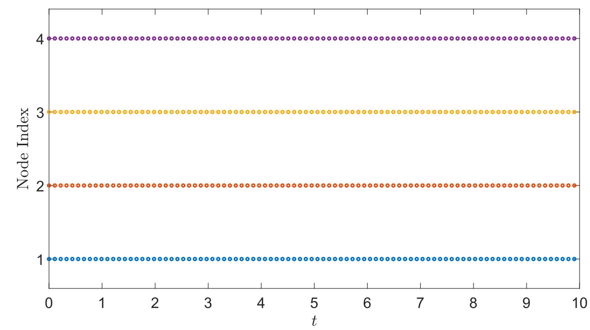

Fig. 9. Example 3 (switching graphs): The triggered instants of the agents with DATT control.

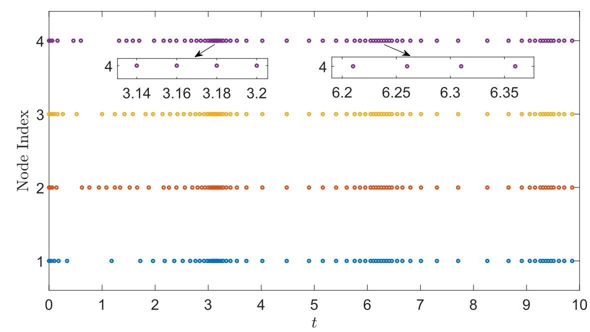

Fig. 10. Example 3 (switching graphs): The triggered instants of the agents with DAET control.

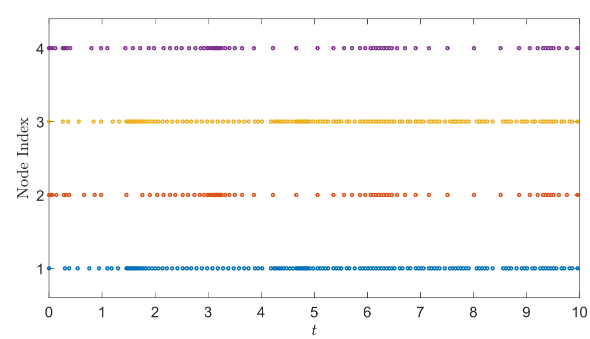

Fig. 11. Example 3 (switching graphs): The triggered instants of the agents with DAET control and heterogeneous threshold functions.

VII. Conclusion

Distributed aperiodic time-triggered and event-triggered (DATT and DAET, respectively) consensus of linear and Lipschitz nonlinear leaderless multiagent systems (MASs)

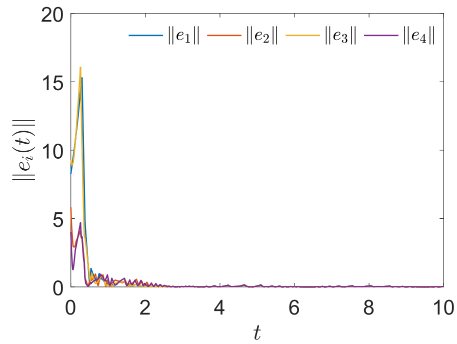

Fig. 12. Example 3 (switching graphs): The consensus error $e_i(t)$ with DAET control and heterogeneous threshold functions.

has been studied. First, we established an explicit upper bound of the maximum sampling interval through an algebraic relation involving the Laplacian eigenvalues, stabilizing gain, and Lipschitz coefficient (for the nonlinear case) for DATT control. Then, we generalized the DATT control to DAET control by leveraging a class of $\mathbf{L}^\infty$ signals. The appealing feature of the proposed method is the scalability to large networks in the sense that the size of the LMIs to be solved is independent of the agent cardinality.

There are several future research directions. First, one may further generalize the framework by considering other types of nonlinear dynamics, e.g., one-sided Lipschitz MASs [41]. In this case, it may need extra analysis on establishing an upper bound of the nonlinear term $\|(\varXi \otimes \mathbf{I}_n)F(x)\|$ in (22). Another interesting topic is to introduce coding-decoding communication protocols (see e.g. [45], [46]) in current sampled-data based control framework, so as to model a more realistic digital communication environment.

Appendix
Proof of Lemma 3

Define $\tilde{W}(t) = W(t) - \frac{\beta_2^s}{\beta_1^s}W(t_s) - \frac{1}{\beta_1^s}Y(t_s),\ t \in [t_s, t_{s+1})$. Then, we have

$$\dot{\tilde{W}}(t) \leq -\beta_1^s \tilde{W}(t), \quad \forall t \in [t_s, t_{s+1}),$$

which guarantees that

$$\dot{\tilde{W}}(t) \leq \tilde{W}(t_s)\ \mathbf{exp}[-\beta_1^s(t-t_s)], \quad \forall t \in [t_s, t_{s+1}).$$

Back to the original $W(t)$, the above equation indicates that

$$\begin{aligned}\dot{W}(t) &\leq W(t_s)\ \mathbf{exp}[-\beta_1^s(t-t_s)] \\ &\quad + (\beta_2^s W(t_s) + Y(t_s)) \int_{t_s}^{t} \mathbf{exp}[-\beta_1^s(t-\tau)]\ \mathbf{d}\tau \\ &= W(t_s)\left(\frac{\beta_2^s}{\beta_1^s}(1 - \mathbf{exp}[-\beta_1^s(t-t_s)]) + \mathbf{exp}[-\beta_1^s(t-t_s)]\right) \\ &\quad + Y(t_s)\int_{t_s}^{t} \mathbf{exp}[-\beta_1^s(t-\tau)]\ \mathbf{d}\tau \quad \forall t \in [t_s, t_{s+1}).\end{aligned}$$

Then, due to the continuity of $W(t)$, there holds

$$W(t_{s+1}) = \lim_{t \to t_{s+1}^-} W(t) \leq \varrho_s W(t_s) + \varsigma_s Y(t_s),$$

where $\varrho_s = \frac{\beta_2^s}{\beta_1^s}(1 - \mathbf{exp}[-\beta_1^s T_s]) + \mathbf{exp}[-\beta_1^s T_s]$, and $\varsigma_s = \int_{t_s}^{t_{s+1}} \mathbf{exp}[-\beta_1^s(t_{s+1}-\tau)]\ \mathbf{d}\tau$. Now, provided $T_s \geq h > 0$ and $\beta_1^s > \beta_2^s > 0$, it is clear that $0 < \varrho_s < 1$, and $0 < \varsigma_s < \frac{1}{\beta_1^s}$. Note that $W(t)$ and $Y(t)$ are both nonnegative, we have

$$W(t_{s+1}) \leq \varrho_s W(t_s) + \varsigma_s Y(t_s) \ < \ W(t_s) + \frac{1}{\beta_1^s}Y(t_s).$$

Since $t_s \to \infty$ as $s \to \infty$, the condition $\lim_{t\to\infty} Y(t) = 0$ guarantees that $\forall \varepsilon > 0$ and $\forall p \in \mathbb{N}$, $\exists\ s^* \in \mathbb{N}$, such that $\forall s > s^*$, there holds $Y(t_s) < \frac{\beta_1^s}{p}\varepsilon$. Thus,

$$W(t_{s+1}) \ < \ W(t_s) + \frac{\varepsilon}{p} \quad \forall s > s^* \tag{36}$$

which implies that

$$|W(t_{s+p}) - W(t_s)| \ < \ \varepsilon \quad \forall s > s^*.$$

The above analysis is the Cauchy's convergence criterion for the series $W(t_s)$, which proves the existence of a limit $W(\infty)$. Taking the limit of s to infinity in (36) gives

$$W(\infty) \ < \ W(\infty) + \frac{\varepsilon}{p}.$$

As ε can be arbitrarily small and p arbitrarily large, we conclude that $W(\infty) = 0$. The proof is completed.

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